Continued Fractions

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The general problem we want to look at

1 Warmup Problems related to the topic

Problem 9.1 A common typesetting error produces $x - 1$ instead of x^{-1}. For what real number or numbers are these two expressions equivalent?

Problem 9.2 If I want to round $\sqrt{15} \approx 3.8729833462$), to the nearest tenth, how close can I come? Or the nearest hundredth? or to the nearest the nearest $1/N$ where N is any positive integer? Do some values of N work better than others? What ratio of integers M/N is closest to $\sqrt{15}$ if we insist that N be at most 10?

Problem 9.3 (British Math Olympiad #3 1987B) Find a pair of integers r and s such that $0 < s < 200$ and

$$\frac{45}{61} < \frac{r}{s} < \frac{59}{80}$$

and prove there is only one such pair r and s.

(hint: One way to prove uniqueness of the solution relies, in part, on an idea we have used in looking at continued fractions. 45/61 and 59/80 are really as close together as they can be, given their denominators . . .)

Problem 9.4 A sequence is formed from the recurrence relation $a_{n+2} = 5a_{n+1} + a_n$. (For example, one such sequence is $1, 3, 16, 83, 431, \ldots$, another is $41, -8, 1, -3, -14, \ldots$) Show that all such sequences may be written in the form $M\alpha^n + N\beta^n$ for some specific values of α, β, M, and N (α and β will be the same for every such sequence, M and N will depend upon the specific sequence).

Problem 9.5 In the game of pool one is given balls arranged in a square tray, but when one ball is used as the cue ball, the others can be racked in a triangle. How many balls could there be?

Problem 9.6 A famous puzzle: On a street in Louvain, Belgium, the houses are numbered consecutively, starting from 1. There are more than 50 houses on the street, but fewer than 500. Find the number of the house for which the sum of all the house numbers less than it equals the sum of the number of houses greater than it. [more generally, find all solutions with no restrictions on the number of houses on the street]. (There's a story involving Ramanujan solving the general problem instantly)

Problem 9.7 A famous problem posed by Archimedes in a letter to Eratosthenes (look up "Archimedes' Cattle Problem" in Wikipedia for details – incidentally, we have an expert on Archimedes scheduled to appear at the math circle in the spring) leads to solving the equation:

$$x^2 - 4729494y^2 = 1$$

where x and y are nonzero integers. How can we solve such an equation – how do we know it even has a solution? (Actually, that may be too much for us to solve. Maybe try solving (in positive integers):

$$x^2 - 57y^2 = 1$$

Problem 9.8 (off topic) **(British Math Olympiad #1 1987A)** (a) Find, with proof, all integer solutions of

$$a^3 + b^3 = 9$$

(b) find with proof, all integer solutions of

$$35x^3 + 66x^2y + 42xy^2 + 9y^3 = 9$$

(hint: The second problem is connected to the first. Don't forget possible solutions involving negative integers for the first one – not that there are any, but how do you know?)

Problem 9.9 (offtopic) **(Canadian Math Olympiad #2 2005)** Let (a, b, c) be a Pythagorean triple, *i.e.*, a triplet of positive integers with $a^2 + b^2 = c^2$.

a) Prove that $(c/a + c/b)^2 > 8$.

b) Prove that there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.

2 The details (outline)

Most of the details are *not* in the handout, we'll go over them during the session,

1. Rational Approximations

   Theorem *(Dirichlet Approximation Theorem) For any positive real number α, and any positive integer n, there are positive integers p and q with $q < n$ for which*

   $$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qn}\frac{1}{q^2}$$

2. What is a continued fraction?

   Simple continue fraction, general continued fraction, finite or infinite, terms may be integers or other things.

3. notation

   $$[a_0, a_1, a_2, a_3, a_4] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}}}$$

4. computing a finite simple continued fraction

   Problem 9.10 Evaluate $[1, 2, 3, 4, 5]$.

5. convergents of a continued fraction

   $$[a_0, a_1, \ldots, a_k] = \frac{p_k}{q_k}$$

   Problem 9.11 Find the first six convergents for the continued fractions $[1, 1, 1, 1, 1, 1]$ and $[2, 1, 2, 1, 2, 1]$

6. useful recurrence relations The inductive definition for continued fractions is that $[a_0] = a_0$, $[a_0, a_1] = a_0 + \frac{1}{a_1}$, and

$$\begin{aligned} [a_0, a_1, \ldots a_n, a_{n+1}] &= [a_0, a_1, \ldots, a_n + 1/a_{n+1}] \\ &= [a_0, a_1, \ldots, [a_n, a_{n+1}]] \end{aligned}$$

   You should verify this yourself – it really is the same thing as the big fraction! And nothing so far depends on the a_k terms being integers, this is all just algebra. Of course, we can't divide by 0, but otherwise, everything works just like you'd expect.

   You can also verify algebraically that $[a_0, a_1, \ldots a_n, a_{n+1}] = [a_0, [a_1, \ldots, a_n, a_{n+1}]]$. (This could be used to give a different inductive definition of continued fractions).

7. recurrence relation for convergents & consequences

   Problem 9.12 What is $\dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}}$?

8. connection to Euclidean algorithm

   Problem 9.13 Find the simple continued fractions expansion for 81/35 and 277/101 and use these to find integers m and n for which $81m + 35n = 1$ and $277m + 101n = 1$

   Problem 9.14 Find the continued fraction expansions for $\dfrac{10! + 1}{11! + 1}$ and $\dfrac{3^7 - 1}{3^8 - 1}$.

9. infinite simple continued fractions (with integer coefficients) always converge

10. continued fractions that converge to any positive real number

    Problem 9.15 Evaluate the real number to which the following continued fraction converges:

$$[5, 2, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, \ldots]$$

    (I want an answer with a square root in it, not a decimal).

11. Every infinite continued fraction converges to an irrational number; Every irrational number has an infinite continued fraction expansion that converges to it.

12. Every periodic (repeating) continued fraction converges to an irrational root of a quadratic equation. Also every irrational root of a quadratic equation has a continued fraction expansion that is (eventually) periodic.

    Problem 9.16 (**extra**) Find the continued fraction expansion for $\sqrt{20}$

13. how good is a continued fraction approximation?

14. Continued fractions and Pell's equation of form

$$x^2 - dy^2 = 1$$

    We can also apply continued fractions to Pell's equation $x^2 - dy^2 = 1$. (d is NOT a perfect square here) In particular, we can show that all solutions for x and y in positive integers come from the convergents of continued fractions for $\sqrt{d}$, where $x = p_n$ and $y = q_n$.

In fact, for every convergent p_n, q_n for $\sqrt{d}$, we must have $|p_n^2 - dq_n^2| \leq \lceil 2\sqrt{d} \rceil$, where $\lceil x \rceil$ stands for "*the least integer greater than or equal to* x".

And if you have any two solutions, $a^2 - b^2\sqrt{d} = u^2 - v^2\sqrt{d}$, then if you multiply out $(a + b\sqrt{d})(u + v\sqrt{d}) = x + y\sqrt{d}$, i.e. $x = au + bvd$, $y = av + bu$, you can verify that $x^2 - y^2 = 1$.

Problem 9.17 Find 2 solutions in positive integers for $x^2 - 6y^2 = 1$.

15. (if time) Louisville theorem.
16. (if time) Continued fractions and primality testing
17. (if time) generalized continued fractions

3 Upshot

Theorem *If α is any irrational number and p_n,q_n and p_{n+1}, q_{n+1} are consecutive convergents of the continued fraction approximation for α, then*

(i) *if n is even, $\frac{p_n}{q_n} < \alpha < \frac{p_{n+1}}{q_{n+1}}$, if n is odd, $\frac{p_{n+1}}{q_{n+1}} < \alpha < \frac{p_n}{q_n}$.*

(ii) *For all n, $|\alpha - \frac{p_n}{q_n}| < \frac{1}{(q_n)^2}$ (could even improve this somewhat!)*

(iii) *For any integer $q < q_n$ and any integer p, $|\alpha - \frac{p_n}{q_n}| < |\alpha - \frac{p}{q}|$*

This could be restated as: **(i)** the convergents alternate between being greater than and less than the number they are approximating, **(ii)** they are *very* good rational approximations and **(iii)** they are the best rational approximations of any rational approximations with the same or smaller denominator.

Theorem *(Liouville, 1844) If $f(x)$ is an nth degree polynomial with integer coefficients, and α is a real root of that polynomial, then there is a number $\kappa > 0$ such that for all rational numbers $\frac{p}{q} \neq \alpha$,*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{\kappa}{q^n}$$

Theorem *(Thue-Siegel-Roth, 1909, 1921, 1955) If $f(x)$ is an nth degree polynomial with integer coefficients, and α is a real root of that polynomial, then there is a number $\kappa > 0$ such that for all rational numbers $\frac{p}{q} \neq \alpha$,*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{\kappa}{q^{2+\epsilon}}$$