San Jose Math Circle
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Mathematical Induction

1. (Vandervelde, *Bridge to Higher* Mathematics, p. 120): "Last One Standing":

Choose four people to sit in a row. Orient yourselves so that one person is in the front and another is in the back. The person in the rear can see the other three, while the person in front can't see anyone else.

   a. All individuals should be seated initially. The object is to arrange for the last person in line to be standing, while all others are seated.
   b. The person in front can stand or sit at will.
   c. The other three participants can move according to the following rule: a person may change state (by standing up or sitting down) if the person immediately in front of them is standing, while all others in front of them are seated. Otherwise they are locked in position and may not move. (The status of people behind them does not matter.)

   Once you have solved the puzzle with four people, try it with three people or with five people. How many moves are required to finish the game? How could the answer for five people have been figured from the answer for four people? Predict how many moves it will take for six people to finish the game.

2. (Fomin, p. 77)
   a. Can you tile a 16x16 grid with L-shaped trominoes (a L-shaped tromino is a 2x2 grid with one square missing)?
   b. What about a 16x16 grid with a square removed? Does it depend on which square we remove?
   c. Generalize this problem and prove.
   d. For what values of n can any grid formed by removing one cell from an n by n grid be tiled with L-shaped trominoes?
3. (Fomin, p. 84) How many regions can be created by n straight lines dividing a plane?
4. (Fomin, p. 87-88; Stankova, p. 109) Identities: Conjecture and prove closed-form (only addition, subtraction, multiplication, division, exponentiation…no infinite sums or "dot dot dot") representations of the following:
   a. 1 + 2 + … + n
   b. 1 + 3 + … + (2n-1)
   c. 1 + 4 + 9 + … + n^2
   d. 1/(1 * 2) + 1/(2 * 3) + 1/(3 * 4) + … + 1/(n*(n+1))

   Try to think of as many different proofs as you can. Some proofs allow you to visualize the sums geometrically and may provide more big-picture insight than mathematical induction proofs. Try to find where the inductive step appears in the visual proof.
5. (Zeitz, p. 46) The plane is divided into regions by some number of straight lines. How many colors are required to color the regions such that adjacent regions are never the same color?
6. Tower of Hanoi (Stankova, p. 121): This puzzle consists of three pegs and n disks, each of a different diameter. The starting position has the disks stacked on one peg, from smallest at the top to largest at the bottom. The task is to move the pile of disks from one peg to another while obeying the rules:
   a. Only one disk at a time may be moved from one peg to another.
   b. No disk may ever be placed over a smaller disk.

Prove that this is always possible.
How many moves are required to move a stack of *n* disks from one peg to another?

7. Divisibility Problems. These may have multiple proofs.
    a. (Fomin, p. 82) Prove that the number 111 … 11 (243 ones) is divisible by 243.
    b. (Stankova, p. 112) Prove that 2^(n+2) + 7^n is divisible by 5 for all n >= 1
    c. (USAMO 2003) Prove that there exists an *n*-digit number, all of whose digits are odd, that is divisible by 5^n.
8. Strong Induction
    a. (Vandervelde, *Bridge to Higher Mathematics*, p. 124) Suppose a professor gives *n* copies of a practice exam to the most responsible member of her class of *n* students, along with the following unusual instructions: At any point a student in possession of an even number of papers may give half of them to a single other student. However, a student with an odd number of papers may only give one paper to another student. Prove that the students can eventually distribute all the papers so that each member of the class receives a single copy, regardless of the class size.
    b. (Stankova, p. 116) Prove that every integer > 1 is a product of primes.
    c. (Fomin, p. 91) Suppose that x + 1/x is an integer. Prove that x^n + 1/x^n is an integer for any positive integer n.
    d. (Benjamin, Chapter 1) How many ways are there to tile a 2 x *n* board with dominoes?
9. (Vandervelde, *Bridge to Higher* Mathematics, p. 126) *n* people line up in a row. In how many moves can they completely reverse their order, where a move consists of two adjacent people trading places?
10. Inequalities
    a. (Stankova, p. 111) Prove that n < 2^n for all n >= 1.
    b. Find all integers n >= 1 such that n^3 > 2^n.

References

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