Monthly Contest 1
Due October 29, 2014

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problem's solution (not the total pages of all the solutions).

DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems.

You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may view any book or website (including forums) as long as you do not communicate the problem in any sort of manner (e.g. posting the problem statement or variations on a forum).

Problems

1. There are two unit circles which have centers spaced 1 unit away from each other. Find the area of the region containing all points which are no further than 1 unit from either circle.

2. Alfred owns a very unusual six-sided die. While the numbers on each of the faces is normal (1 through 6), the die is weighted in such a way that the probability of a number x appearing on top is directly proportional to x^2. If he rolls the die twice, what is the probability that the sum of the two numbers he rolls is equal to 7?

3. Find the minimum possible value of $\frac{36}{x^2} + x^2$ for $x \neq 0$.

4. Prove that there exist an infinite number of primes p such that $p \equiv 3 \pmod 4$.

5. You have an electronic balance scale and 16 coins. All of these coins are the same weight, except for one, and you do not know whether this coin is heavier or lighter than the rest. When you place coins on the scale, it will say "X" or "O" depending on whether the weight on both sides match or not. However, you do not remember whether "X" means that the sides balance or if "O" does. What is the least number of weightings you must do to determine which of the coins is different? You must prove that this number is the smallest possible to receive full credit.