Monthly Contest 5 Solutions

Here are the solutions for the fifth and last monthly contest of the 2013-2014 school year. The solutions listed below are not the only solutions. I encourage you to explore each problem and see if you can discover other approaches to the problems. Perhaps you may even find a solution more elegant than the ones listed!

Problem 1 Let a_1, a_2, and a_3 be positive numbers such that $a_1 + a_2 + a_3 = 1$. Show that $a_1a_2a_3 \leq \frac{1}{27}$ and find when equality occurs.

Solution From AM-GM, we know that $\dfrac{1}{3} = \dfrac{a_1 + a_2 + a_3}{3} \geq \sqrt[3]{a_1a_2a_3}$. Cubing both sides, we get that

$$\frac{1}{27} = (\frac{1}{3})^3 \geq (\sqrt[3]{a_1a_2a_3})^3 = a_1a_2a_3$$

Problem 2 Show that $(1 + 2\sqrt{2} + 3\sqrt{3} + \cdots + (n-1)\sqrt{n-1} + n\sqrt{n})^2 \leq (\dfrac{n(n+1)}{2})(\dfrac{n(n+1)(2n+1)}{6})$ for any positive integer n. Find when equality occurs.

Solution Cauchy-Schwartz gives us that

$$(1^2 + 2^2 + \cdots + n^2)((\sqrt{1})^2 + (\sqrt{2})^2 + \cdots + (\sqrt{n})^2) \geq (1 + 2\sqrt{2} + \cdots + n\sqrt{n})$$

Using the fact that $1 + 2 + \cdots n = \dfrac{n(n+1)}{2}$ and $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$, we get that

$$(1 + 2\sqrt{2} + 3\sqrt{3} + \cdots + (n-1)\sqrt{n-1} + n\sqrt{n})^2 \leq (\frac{n(n+1)}{2})(\frac{n(n+1)(2n+1)}{6})$$

Problem 3 Prove the Harmonic Mean-Geometric Mean (HM-GM) Inequality. That is, show that for positive numbers
a_1, a_2, a_3, $\cdots$, a_{n-1}, a_n, the following holds true:

$$\frac{1}{\dfrac{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \cdots + \dfrac{1}{a_{n-1}} + \dfrac{1}{a_n}}{n}} \leq \sqrt[n]{a_1a_2a_3\cdots a_{n-1}a_n}$$

Also find when equality occurs.

Solution Note that by AM-GM, we have

$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1}\cdot\frac{1}{a_2}\cdots\frac{1}{a_{n-1}}\cdot\frac{1}{a_n}}$$

By taking the reciprocal of both sides, we get

$$\frac{1}{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_{n-1}}+\frac{1}{a_n}}{n}} \leq \sqrt[n]{a_1a_2a_3\cdots a_{n-1}a_n}$$

Problem 4 Let p represent the perimeter of $\triangle ABC$ and let R and S represent the circumradius and area, respectively. Prove that $p^3 \geq 108RS$ and find when equality occurs.

Solution Let a, b, and c represent the side lengths of the triangle. Then by AM-GM, $p = a+b+c \geq 3\sqrt[3]{abc}$. We can cube both sides to get $p^3 \geq 27abc$. We also know that $4RS = abc$ so by combining these, we get $p^3 \geq 27abc = 27(4RS) = 108RS$.

Problem 5 Prove that for nonnegative numbers a_1, a_2, a_3, $\cdots$, a_{n-1}, a_n,

$$a_1^2+a_2^2+a_3^2+\cdots+a_{n-1}^2+a_n^2 \geq (a_1+a_2+a_3+\cdots+a_{n-1}+a_n)\sqrt[n]{a_1a_2a_3\cdots a_{n-1}a_n}$$

and find when equality occurs.

Solution By Cauchy-Schwartz, we have that

$$(a_1^2+a_2^2+\cdots+a_n^2)\underbrace{(1^2+1^2+\cdots+1^2)}_{\text{Repeated } n \text{ times}} \geq (a_1+a_2+\cdots+a_n)^2$$

By rearranging and using AM-GM, we get

$$a_1^2+a_2^2+\cdots+a_n^2 \geq (a_1+a_2+\cdots+a_n)\cdot\frac{(a_1+a_2+\cdots a_n)}{n} \geq (a_1+a_2+\cdots+a_n)\sqrt[n]{a_1a_2\cdots a_n}$$