Monthly Contest 5
Due April 23, 2014

Instructions

This contest consists of 6 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problem's solution (not the total pages of all the solutions). DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems.

You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may consult any book that you wish.

Additional Notes

This contest is different from previous monthly contest. For this test, all of the problems are focused on a single theme: inequalities! Each of the problems can be solved using the AM-GM inequality and the Cauchy-Schwartz Inequality. In case you don't remember, they are listed below.

The Arithmetic Mean-Geometric Mean (AM-GM) Inequality states that for nonnegative numbers $a_1, a_2, a_3, \cdots, a_{n-1}, a_n$, the following holds true:

$$\frac{a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \cdots a_{n-1} a_n}$$

Equality holds if and only if $a_1 = a_2 = a_3 = \cdots = a_{n-1} = a_n$.

The Cauchy-Schwartz Inequality states that

$$(a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \cdots + b_{n-1}^2 + b_n^2) \geq (a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_{n-1}b_{n-1} + a_nb_n)^2$$

for all real a_i and b_i. Equality holds if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3} = \cdots = \dfrac{a_{n-1}}{b_{n-1}} = \dfrac{a_n}{b_n}$.

If you have an alternate proof which does not use these inequalities, feel free to use that instead. Math isn't limited to just a few theorems so feel free to research other inequalities or methods to solve the problems.

Problems

1. Let a_1, a_2, and a_3 be positive numbers such that $a_1 + a_2 + a_3 = 1$. Show that $a_1a_2a_3 \leq \frac{1}{27}$ and find when equality occurs.

2. Show that $(1 + 2\sqrt{2} + 3\sqrt{3} + \cdots + (n-1)\sqrt{n-1} + n\sqrt{n})^2 \leq (\dfrac{n(n+1)}{2})(\dfrac{n(n+1)(2n+1)}{6})$ for any positive integer n. Find when equality occurs.

3. Prove the Harmonic Mean-Geometric Mean (HM-GM) Inequality. That is, show that for positive numbers $a_1, a_2, a_3, \cdots, a_{n-1}, a_n$, the following holds true:

$$\frac{1}{\dfrac{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \cdots + \dfrac{1}{a_{n-1}} + \dfrac{1}{a_n}}{n}} \leq \sqrt[n]{a_1a_2a_3 \cdots a_{n-1}a_n}$$

Also find when equality occurs.

4. Let p represent the perimeter of $\triangle ABC$ and let R and S represent the circumradius and area, respectively. Prove that $p^3 \geq 108RS$ and find when equality occurs.

5. Prove that for nonnegative numbers $a_1, a_2, a_3, \cdots, a_{n-1}, a_n$,

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2 \geq (a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n)\sqrt[n]{a_1a_2a_3 \cdots a_{n-1}a_n}$$

and find when equality occurs.

6. Please write any suggestions for or comments about the monthly contests of this school year. Please tell us which contests were too hard, which were too easy, which ones were too boring, which ones were too interesting (hopefully this one isn't the case), or anything else which can help us write up better problems next year. Also feel free to tell us which contests you liked the best. It's our first year doing this so any feedback is greatly appreciated. We'll also give you 7 free points for answering this question!