Monthly Contest 2 Solutions

Here are the solutions for the second monthly contest of the 2013-2014 school year. The solutions listed below are not the only solutions. I encourage you to explore each problem and see if you can discover other approaches to the problems. Perhaps you may even find a solution more elegant than the ones listed!

Problem 1 Show that in any right triangle with integer side lengths, an odd number of sides have an even length.

Solution First, we use the fact that x and x^2 have the same parity. That is, x is even if and only if x^2 is even. Now, there are three cases: the two legs have odd side length, the two legs have even side length, or one leg has even side length and one has odd side length. The first and second cases require even hypotenuses and the third case results in an odd one. For all three cases, there is an odd number of sides with an even length.

Note: Using the fact that x^2 can only be 0 or 1 (mod 4), one may actually prove that the first case is not possible.

Problem 2 Find all integer solutions to the inequality

$$\frac{x-1}{3} < \frac{5}{7} < \frac{x+4}{5}$$

Solution In order to satisfy the problem, both the inequalities $\frac{x-1}{3} < \frac{5}{7}$ and $\frac{5}{7} < \frac{x+4}{5}$ must be satisfied. For the first one, we can cross-multiply to get $7x - 7 < 15 \Rightarrow x \leq 3$. Similarly, we get $25 < 7x + 28 \Rightarrow 0 \leq x$. Combining the two, we find that x must be equal to 0, 1, 2, or 3.

Problem 3 When walking up stairs, Joe can walk up 1, 2, or 3 stairs in a step. For example, he can go up 6 steps by moving up 2 steps, 3 steps, and then 1 step or 1 step, 2 steps, 2 steps, and then 1 step. Let a_n represent the number of ways he can climb up n stairs. Show that the following relation holds:

$$a_n = 2a_{n-2} + 2a_{n-3} + a_{n-4}$$

Solution One may note that $a_0 = 1$, $a_1 = 1$, and $a_2 = 2$. Now for any a_n with $n \geq 3$, we can split it into three cases: either the last step moved up 1 stair, 2 stairs, or 3 stairs. If Joe moved up 1 stair, then there are a_{n-1} ways which he could have accomplished this as there are a_{n-1} ways to move up $n-1$ stairs and then move that final one. Similarly, there are a_{n-2} ways that his last step moved 2 stairs and a_{n-3} ways his last step moved 3 stairs. Thus, the recursion we get is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. We can substitute this equation for a_{n-1} to get

$$a_n = (a_{n-2} + a_{n-3} + a_{n-4}) + a_{n-2} + a_{n-3} = 2a_{n-2} + 2a_{n-3} + a_{n-4}$$

Problem 4 On parallelogram ABCD, point E lies on the midpoint of $\overline{AD}$. Point F is drawn such that $\overline{BF}$ is perpendicular to $\overline{CE}$. Show that $\triangle ABF$ is isosceles.

Note: The version of the problem originally given was incorrect. Point F should be on $\overline{CE}$. This information was missing and without this the problem was not solvable. The correct version of this problem will be given in Monthly Contest 3.

Problem 5 Each of the numbers $x_1, x_2, \cdots, x_n$ equals 1 or -1 and

$$\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = 0$$

where $x_{n+i} = x_i$. Prove that $4|n$.

Solution 1 One may consider

$$\prod_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = \prod_{i=1}^{n} x_i^4$$

Note that $x_i^4 = 1$ for any i so the right hand side is equal to 1. Also, from the original sum, the number of $x_i x_{i+1} x_{i+2} x_{i+3}$ which are positive is equal to the number which are negative. For the product of these products to be 1, there must an even number of negative products. Since there are n products and the number of products is equal to twice an even number, n must be divisible by 4.

Solution 2 If the sign of a term x_k is flipped, then the four products with the term will also have their sign flipped. If all four products had the same sign, then flipping them will change the overall sum by 8. If three of the four products were the same sign, then the sum will change by 4. If only two products were the same, then flipping will not change the sum. One may notice that any of these transformations do not affect the sum modulo 4. If we let all of the x_i equal 1, then their sum must be a multiple of 4 as flipping a certain subset of them will result in 0. Since this sum is equal to the number of terms, 4 must also divide n.