Monthly Contest 2
Due December 4, 2013

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of problem's solution (not the total pages of all the solutions). DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems.

You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may consult any book that you wish.

Problems

1. Show that in any right triangle with integer side lengths, an odd number of sides have an even length.

2. Find all integer solutions to the inequality

$$\frac{x-1}{3} < \frac{5}{7} < \frac{x+4}{5}$$

3. When walking up stairs, Joe can walk up 1, 2, or 3 stairs in a step. For example, he can go up 6 steps by moving up 2 steps, 3 steps, and then 1 step or 1 step, 2 steps, 2 steps, and then 1 step. Let a_n represent the number of ways he can climb up n stairs. Show that the following relation holds:

$$a_n = 2a_{n-2} + 2a_{n-3} + a_{n-4}$$

4. On parallelogram ABCD, point E lies on the midpoint of $\overline{AD}$. Point F is drawn such that $\overline{BF}$ is perpendicular to $\overline{CE}$. Show that $\triangle ABF$ is isosceles.

5. Each of the numbers $x_1, x_2, \cdots, x_n$ equals 1 or -1 and

$$\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = 0$$

where $x_{n+i} = x_i$. Prove that $4|n$.