Monthly Contest 1 Solutions

Here are the solutions for the first monthly contest of the 2013-2014 school year. The solutions listed below are not the only solutions. I encourage you to explore each problem and see if you can discover other approaches to the problems. Perhaps you may even find a solution more elegant than the ones listed!

Problem 1 Show that if we pick five lattice points (points (x,y) on the coordinate plane such that x and y are integers), two of them must have a midpoint which is also a lattice point.

Solution The formula for the midpoint of two points (x_1,y_1) and (x_2,y_2) is $(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2})$. In order for the midpoint to be integer, both x_1 and x_2 must be of the same parity and both y_1 and y_2 must be of the same parity. We may consider the following possible pairs of parity of the x, y coordinates: (even,even), (even,odd), (odd,even), and (odd,odd). We may note that if two of the points have the pair, then we are done as the sum of their x and y coordinates are both even and the midpoint will have integer coordinates. Since there are 5 points and 4 possible pairs, the Pigeonhole Principle states that two points must belong to the same pair and two points have a lattice point for their midpoint.

Problem 2 Let the sequence of integers $a_0,a_1,a_2,\ldots,a_n$ be called interesting if it has the property that for every integer $0 \leq k \leq n$, the number of times k appears in the sequence is a_k. For example, $a_0 = 1, a_1 = 2, a_2 = 1, a_3 = 0$ would be called an interesting sequence. Given an interesting sequence $a_0,a_1,a_2,\ldots,a_m$, find $\sum_{i=0}^{m} a_i$.

Solution Note that a_0 represents the number of 0's in the sequence, a_1 represents the number of 1's in the sequence, and so on. This gives us that $a_0 + a_1 + \cdots + a_{n-1} + a_m$ represents the number of values in the sequence between 0 and n. Clearly, $0 \leq a_i \leq m+1$ for all $0 \leq i \leq m$ as the number of i's in the sequence is nonnegative and cannot be more than the number of values in the sequence, which is $m - 0 + 1 = m+1$. We now show that $a_i \neq m+1$ for all $0 \leq i \leq m$. If $a_i = m+1$, then this would mean that all $m+1$ members of the sequence equal i. Then all members of the sequence must be equal to i, but that means $a_i = i \neq m+1$. A Contradiction! Thus, all members of the sequence are between 0 and m so $\sum_{i=0}^{m} a_i = m+1$

Problem 3 Alice and Betty are playing a game. First, they pick two positive integers k and n. They write down the number n on a whiteboard. On a players turn, she may replace n with the number $n - k^m$ where m is a nonnegative integer and $n \geq k^m$. A player loses if she cannot make a move. Let Alice make the first move. For which n does Betty have the winning strategy?

Solution We first note that if k is odd, then k^m must be odd as well. This means that for Betty to win (game ends in an even number of moves), n must be even as well. If k is even, then we have some more work. First, we note that for all $n < k$, we have that Betty wins if n is even as the only possible move is to replace n with $n-1$. Because we know the winning states (states where the person currently makeing a move has a winning strategy) for $n < k$ (which are all n such that $n \equiv 2i-1 \pmod{k-1}$ and $n \equiv 0 \pmod{k-1}$ for some positive integer $i < \frac{k-1}{2}$), we can show that Betty wins if $n \equiv 2i \pmod{k-1}$ for some positive integer $i < \frac{k-1}{2}$. We prove our result using strong induction by assuming that all numbers up to $n-1$ follow this rule. Because of the fact that $k^m \equiv 1 \pmod{k-1}$, every winning state is only a winning state if $n-1 \equiv 2i \pmod{k-1}$ for some positive integer $i < \frac{k-1}{2}$. As a result, we show for all n that the only n for which Betty has the winning strategy for odd k are $n \equiv 2i \pmod{k-1}$ for positive integers $i < \frac{k-1}{2}$. Thus, combining our two results for even and odd k, we get that Betty has the winning strategy if and only if $n \equiv 2i \pmod{k-1}$ for $i < \frac{k-1}{2}$ for some positive integer i. If k is odd, we also have the additional solution $n \equiv 0 \pmod{k-1}$.

Problem 4 On acute triangle $\triangle ABC$, there is a point P on $\overline{BC}$. Find points X and Y on $\overline{AB}$ and $\overline{AC}$ respectively such that the perimeter of $\triangle PXY$ is minimized.

Solution We reflect P across $\overline{AB}$ and $\overline{AC}$ to get the points S and T respectively. Note that $\overline{AP} = \overline{AS}$, $\overline{AX} = \overline{AX}$, and $\angle BAP = \angle BAS$. This means that $\triangle ABP \cong \triangle ABS$ so $\overline{XP} = \overline{SX}$. Similarly, we have that $\overline{YP} = \overline{TY}$. This means that the perimeter of $\triangle PXY$ is equal to $\overline{XP} + \overline{XY} + \overline{YP} = \overline{SX} + \overline{XY} + \overline{YT}$. Also, for a fixed point P, we have that S and T are fixed so the minimum possible perimeter is the minimum length of $\overline{SX} + \overline{XY} + \overline{YT} = \overline{ST}$. Thus, if we take X and Y to be the intersections of $\overline{ST}$ with $\overline{AB}$ and $\overline{AC}$, we are finished.

Problem 5 For reals a, b, c, d, e, we have that $a + b + c + d + e = 1$. Show that

$$a^4 + b^4 + c^4 + d^4 + e^4 \geq \frac{1}{125}$$

and find when equality occurs.

Solution 1 We start out with the Cauchy-Schwartz Inequality:

$$(a^2 + b^2 + c^2 + d^2 + e^2)(1 + 1 + 1 + 1 + 1) \geq (a + b + c + d + e)^2 = 1$$
$$\Rightarrow a^2 + b^2 + c^2 + d^2 + e^2 \geq \tfrac{1}{5}$$

We then do this again:

$$(a^4 + b^4 + c^4 + d^4 + e^4)(1 + 1 + 1 + 1 + 1) \geq (a^2 + b^2 + c^2 + d^2 + e^2)^2 = \tfrac{1}{25}$$
$$\Rightarrow a^4 + b^4 + c^4 + d^4 + e^4 \geq \tfrac{1}{125}$$

Solution 2 We note that $f(x) = x^4$ is a convex function so we can use Jensen's inequality.

$$a^4 + b^4 + c^4 + e^4 \geq 5f(\frac{a + b + c + d + e}{5}) = 5f(\frac{1}{5}) = \frac{1}{125}$$