Monthly Contest 1
Due October 23, 2013

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of problem's solution. DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems.

You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may consult any book that you wish.

Problems

1. Show that if we pick five lattice points (points (x,y) on the coordinate plane such that x and y are integers), two of them must have a midpoint which is also a lattice point.

2. Let the sequence of integers a_0,a_1,a_2,. . . ,a_n be called interesting if it has the property that for every integer $0 \leq k \leq n$, the number of times k appears in the sequence is a_k. For example, $a_0 = 1$,$a_1 = 2$,$a_2 = 1$,$a_3 = 0$ would be called an interesting sequence. Given an interesting sequence a_0,a_1,a_2,. . . ,a_m, find $\sum_{i=0}^{m} a_i$.

3. Alice and Betty are playing a game. First, they pick two positive integers k and n. They write down the number n on a whiteboard. On a players turn, she may replace n with the number $n - k^m$ where m is a nonnegative integer and $n \geq k^m$. A player loses if she cannot make a move. Let Alice make the first move. For which n does Betty have the winning strategy?

4. On acute triangle $\triangle ABC$, there is a point P on $\overline{BC}$. Find points X and Y on $\overline{AB}$ and $\overline{AC}$ respectively such that the perimeter of $\triangle PXY$ is minimized.

5. For reals a, b, c, d, e, we have that $a + b + c + d + e = 1$. Show that

$$a^4 + b^4 + c^4 + d^4 + e^4 \geq \frac{1}{125}$$

and find when equality occurs.