Introduction to Counting VIII

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Today we are going to do some miscellaneous problems to see if we can use all the techniques we've learned so far, and maybe learn some new ones. (Many of these are taken from Dave Patrick's *Introduction to Counting & Probability*, published by Art of Problem Solving.)

Probability

1. A class of 15 students is split into 5 groups of 3.
   (a) What is the probability that Alice and Bob are in the same group?
   (b) What is the probability that Alice, Bob, and Carol are in the same group?

2. A class of 20 students is split into 5 groups of 4.
   (a) What is the probability that Alice and Bob are in the same group?
   (b) What is the probability that Alice, Bob, and Carol are in the same group?

3. Six points are placed evenly around a circle and labeled A, B, C, D, E, and F at random. What is the probability that triangles ABC and DEF do not overlap?

Pascal

4. In the nth row of Pascal's triangle, which has n+1 numbers, multiply the zeroth number by 0, the first number by 1, ..., the nth number by n, and then sum the results. What pattern do you notice? How can you prove it?

   Is it easier or harder if you multiply by 1, 2, 4, 8, ... instead of 0, 1, 2, 3, ...?

5. What is the sum of the squares of the numbers in a row of Pascal's triangle?

6. How can you predict whether a given number in Pascal's triangle will be odd or even? For instance, is the 37th number in the 142nd row odd or even? How about the 38th number in the 143rd row?

Miscellaneous

7. A convex polyhedron is formed such that every vertex has a regular pentagon, triangle, square, and triangle around it in that order. How many vertices does it have? How many of each of the regular polygons does it have for its faces? How many edges does it have?

8. Each of two teams has 10 players numbered 1 through 10. The two players with number 1 play each other, and the loser is eliminated. The next game is between the surviving number 1 and player 2 on the other team. Each game, the loser is eliminated and then the next game is between the lowest-numbered people on each team. How many different sequences of games are possible?

9. How many lists of the numbers 1 through 8 have the property that each digit is either bigger than all the numbers preceding it or smaller than all the numbers preceding it? For instance, 4, 5, 6, 3, 2, 7, 1, 8 is one such list.

10. In an *n* by *n* square grid of dots, how many squares are there whose dots are the vertices of the grid? (Hint: with 3 dots on each edge, the answer is 6, not 5.)

11. Is it possible to paint the sides of 27 1x1x1 cubes red, green, and blue, in such a way that the resulting collection of cubes could be put together into a 3x3x3 cube of any one of the three colors?

12. A bag has red, white, blue and green marbles in it. When drawing 4 marbles without replacement, each of the following has the same probability:
    (a) One marble of each color
    (b) One white, one blue, and two red
    (c) One blue and three red
    (d) Four red
    What is the smallest possible number of marbles in the bag?

13. A basketball team has a 2/3 chance of winning each of its games. What is the probability that it wins at least four in a row out of eight games that it plays?