

Introduction to Counting VI

February 8, 2012

Joshua Zucker, joshua.zucker@stanfordalumni.org

The idea for today is to count the number of ways of splitting up a positive integer into a sum of one or more positive integers. We'll see why there are very big differences between counting ordered lists (called *compositions*) and unordered lists (called *partitions*). And we'll discover various relationships by organizing the counting in different ways, and by restricting the counting to cover only certain categories.

Compositions

There are four compositions of 3: 3, 2+1, 1+2, and 1+1+1. Sorting them like this, from largest number to smallest, is one way of making sure you haven't left any out.

1. How many compositions does 4 have? Can we organize a list well enough to determine how many compositions each number has, from 1 through 10? How many compositions should 0 have to fit the pattern?
2. Let's subcategorize these in different ways. What patterns do you notice if you organize the compositions based on how big the first part is?
How many compositions of 10 have first part 4?
3. What patterns do you notice if you organize the compositions based on how many parts there are?
How many compositions of 10 will have 1 part? 9 parts? 4 parts?
4. What if we organize them by how big the largest part is? How many compositions of 10 have largest part 1? Largest part 2? Largest part 3?
Is it easier to count the number with largest part k or with largest part less than or equal to k ?
5. What if we count the compositions with all their parts different? Can we organize all the compositions in this way?

Partitions

There are three partitions of 3: 3, 2+1, and 1+1+1, because now 2+1 and 1+2 are considered the same.

6. How can you organize a table of partitions to see how many partitions there are for each number 1 through 10? Is this harder or easier than the composition question?
7. One way to organize your table is based on the smallest part. How many partitions of 5 have a smallest part of 2?

In general, can you find a pattern for how many partitions have smallest part 2? Or maybe it's easier to count how many partitions have no parts smaller than 2?

8. How many partitions of 5 into 2 parts are there? Can you generalize this, so that you can predict how many partitions of 10 into 4 parts there will be?
9. How many partitions of a number are there if all the parts must be even?
10. How many partitions are there when each part occurs twice? Is it easier to count the situations with at least two copies of each part, or exactly two?

Diagrams

Sometimes it's very helpful to be able to make a picture of whatever problem you're working on. For partitions, one useful picture is the "Ferrers diagram" where you list the parts in decreasing order and use dots to represent the numbers. For example, all the partitions of 4 will look like this:

4	3+1	2+2	2+1+1	1+1+1+1
****	***	**	**	*
	*	**	*	*
			*	*
				*

11. There's a relationship between 4 and 1+1+1+1: these partitions are *conjugates*, or you might say one is the *transpose* of the other, switching the rows and columns of the diagrams. What is the conjugate of 4+2+1 among the partitions of 7?
12. How many ways are there to partition a number into distinct odd parts? For instance, for 4 we see that 3+1 is the only such partition.
13. How many partitions of a number are self-conjugate? For instance, for 4 we see that 2+2 is the only self-conjugate partition.
14. Count the total number of 1s that appear in all the partitions of a number. Compare with the total number of different parts that appear in all the partitions (so for instance, for 4 you have two different parts in 2+1+1 and one more for 1+1+1+1, and so on).
15. Add up the largest part in all the partitions of each number. For instance, for 4 you would compute $4 + 3 + 2 + 2 + 1$.
16. When is the number of partitions into an odd number of different parts the same as the number of partitions into an even number of different parts? For instance, for 4 the only partitions into different parts are 4 and 3+1, so there's one of each.