Introduction to Counting IV

December 14, 2011

Joshua Zucker, joshua.zucker@stanfordalumni.org

Triangles

For each type of numbers described below (choose, subset, cycle, ascent, derangement):

1. Determine a formula for $(n, 0)$.
2. Determine a formula for $(n, 1)$.
3. Determine a formula for $(n, 2)$.
4. Determine a formula for the sum of all the (n, k) numbers (in terms of n).
5. Write out the first four or five rows of the triangle (in the case of choose, named after Pascal because he didn't invent it, and in the case of Stirling numbers of the second kind called that because he talked about them first).
6. Determine a formula for (n, k) in terms of $(n–1, k)$ and $(n–1, k–1)$.

Binomial

Binomial coefficients, or "choose" numbers, count the number of ways of splitting a set of n elements into two subsets, one of size k and one of size n–k. For example, 5 choose 2 is 10 because you can have 12, 13, 14, 15, 23, 24, 25, 34, 35, 45 as the subset of size 2. The usual notation is $\binom{n}{k}$.

Stirling, part 2

Stirling numbers of the second kind, or "subset" numbers, count the number of ways of splitting a set of n elements into k nonempty subsets. For example, *5* subset 2 is 15 because you can have (for the smaller of the two subsets 1, 2, 3, 4, 5, as well as the 10 sets listed above. The usual notation is $\left\{ {n \atop k} \right\}$.

Stirling, part 1

Stirling numbers of the first kind, or "cycle" numbers, count the number of ways of splitting a set of n elements into k nonempty cycles. A cycle is like a subset except the numbers are seated around a circle, so for instance [123] is the same as [231] and [312] but different from [132]. The usual notation is $\begin{bmatrix} n \\ k \end{bmatrix}$.

Eulerian numbers

Eulerian numbers count the number of rearrangements of *n* numbers that have exactly *k* ascents. An ascent is a number in the rearrangement that's greater than the immediately preceding number. For example, 4 "ascent" 2 is 11 because of 1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, 4123. The usual notation is $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$.

Derangements

Derangements count the number of rearrangements of *n* numbers that have exactly *k* items in their original place. The case of $k = 0$ is usually the most interesting. For example, we can see that 4 "derange" 2 is 6 because 1243, 1432, 1324, 4231, 3214, and 2134 are the possible ways to have exactly 2 of the 4 items in their original place. There isn't a usual notation that I know of, but I often see $D(n, k)$.

Miscellaneous

These are counting problems from an old Polya competition sometime in the late 90s.

7. In the first row of spectators at a table tennis tournament, there are 12 seats. In how many ways can the people in that row change seat positions by exactly one seat?

8. In the second row, there are 11 seats. In how many ways can the people in that row change seat position by at most one seat?

9. In the third row, there are 10 seats. In how many ways can the people in that row change seat position by exactly two seats?

10. Ten people sit around a circular table at a restaurant to watch the tournament on TV. In how many ways can they change their seat position by exactly one seat? At most one seat? Exactly two seats?

11. If the 33 fans at the tournament sit in a rectangular block of 3 rows of 11 seats, in how many ways can everyone move by exactly one seat (forward or backward or left or right?)

12. If a group of 16 fans sit in a 4 by 4 square, how many ways are there for them to move exactly two seats in one direction?

13. If the best player always wins, and there are 16 players in a single-elimination match, what's the probability that the loser of the final match is the second-best player? kth best?

14. Skittles come in 5 different colors. What's the probability that, in a random set of 5, there are more yellow Skittles than any other color?

15. How many different flavor combinations could be made from a collection of four of each color skittle? (Assume that 2 green, 1 red, 1 yellow tastes the same as 4 green, 2 red, 2 yellow, since the ratios are the same.)

16. How many different sets of 8 skittles are possible, using 5 colors?

17. If two people rank Skittles preferences at random, what's the probability that there's exactly one pair of colors that they would both be happy to trade with each other?