COMPLEX NUMBERS II - MORE ON THE ALGEBRAIC FORM

1. Let $z_1, z_2 \in \mathbb{C}$. Prove that the number $E = z_1 \cdot \overline{z_2} + \overline{z_1} \cdot z_2$ is a real number.

2. Prove that if $|z_1| = |z_2| = 1$ and $z_1 z_2 \neq -1$, then $\dfrac{z_1 + z_2}{1 + z_1 z_2}$ is a real number.

3. Consider the complex numbers z_1, z_2,..., z_n with
$$|z_1| = |z_2| = \cdots = |z_n| = r > 0.$$
Prove that the number
$$E = \frac{(z_1 + z_2)(z_2 + z_3) \dots (z_{n-1} + z_n)(z_n + z_1)}{z_1 z_2 \dots z_n}$$
is real.

4. Prove that:

(a) $E_1 = (2 + i\sqrt{5})^7 + (2 - i\sqrt{5})^7 \in \mathbb{R}$

(b) $E_2 = \left(\frac{19 + 7i}{9 - i}\right)^n + \left(\frac{20 + 5i}{7 + 6i}\right)^n \in \mathbb{R}$

5. Let z_1, z_2, z_3 be complex numbers such that:

$$|z_1| = |z_2| = |z_3| = r > 0$$

and $z_1 + z_2 + z_3 \neq 0$. Prove that

$$\left|\frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{z_1 + z_2 + z_3}\right| = r.$$

6. Prove the identity:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

for all complex numbers z_1 and z_2.

7. Prove the identity:

$$|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1 + z_2 + z_3|^2$$

for all complex numbers z_1, z_2, and z_3.

8. *(1988 Romanian Math Olympiad, School Competition, 10th grade)* Let $z_1, z_2, z_3 \in \mathbb{C}$ such that $\mathrm{Im}(\overline{z_1} z_2) = \mathrm{Im}(\overline{z_2} z_3) = \mathrm{Im}(\overline{z_3} z_1) \neq 0$. Show that:

$$|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2).$$

9. Prove the following identities:

(a) $|1 + z_1\overline{z_2}|^2 + |z_1 - z_2|^2 = (1 + |z_1|^2)(1 + |z_2|^2)$

(b) $|1 - z_1\overline{z_2}|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$

10. Prove the identity:

$$|z_1 + z_2 + z_3|^2 + |-z_1 + z_2 + z_3|^2 + |z_1 - z_2 + z_3|^2 + |z_1 + z_2 - z_3|^2 = 4(|z_1|^2 + |z_2|^2 + |z_3|^2)$$

for all complex numbers z_1, z_2, and z_3.

11. Let z_1, z_2, z_3 be complex numbers such that

$$z_1 + z_2 + z_3 = 0 \quad \text{and} \quad |z_1| = |z_2| = |z_3| = 1.$$

Prove that

$$z_1^2 + z_2^2 + z_3^2 = 0.$$

12. *(1979 Romanian Math Olympiad, State Competition, 10th grade)* Let z_1, z_2, z_3 be distinct complex numbers such that

$$|z_1| = |z_2| = |z_3| > 0.$$

If $z_1 + z_2 z_3$, $z_2 + z_1 z_3$, and $z_3 + z_1 z_2$ are real numbers, prove that $z_1 z_2 z_3 = 1$.

13. *(1988 Romanian Math Olympiad, School Competition, 10th grade)* If $z_1, z_2, z_3 \in \mathbb{C}$ such that $|z_1| = |z_2| = |z_3| = 1$, show that:

$$(z_1 z_2 + z_2 z_3 + z_3 z_1 + z_1 + z_2 + z_3)^2 = z_1 z_2 z_3 |z_1 z_2 + z_2 z_3 + z_3 z_1 + z_1 + z_2 + z_3|^2$$

14. *(1987 Romanian Math Olympiad, School Competition, 10th grade)* Let $z_1, z_2, z_3 \in \mathbb{C}^\star$.

(a) If $|z_1| = |z_2| = |z_3|$, then

$$z_1 + z_2 + z_3 = 0 \quad \text{if and only if} \quad z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$$

(b) If $z_1 + z_2 + z_3 = z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$, then $|z_1| = |z_2| = |z_3|$.

15. *(1990 Romanian Math Olympiad, School Competition, 10th grade)* Let $z_1, z_2 \in \mathbb{C}$. Show that

$$\max\{|z_1|, |z_2|\} \leq |z_1 + z_2| + \sqrt{|z_1 z_2|}$$

16. Let $z \in \mathbb{C}$. Show that

$$\operatorname{Im}(z) > 0 \Leftrightarrow \left|\frac{z-i}{z+i}\right| < 1.$$

17. Prove that for any complex number z,

$$|z+1| \geq \frac{1}{\sqrt{2}} \quad \text{or} \quad |z^2+1| \geq 1$$

18. Let u, v, w, z be complex numbers such that $|u| \leq 1$, $|v| = 1$, and $w = \dfrac{v(u-z)}{\overline{u} \cdot z - 1}$. Prove that $|w| \leq 1$ if and only if $|z| \leq 1$.

19. Prove that

$$\sqrt{\frac{7}{2}} \leq |1+z| + |1-z+z^2| \leq 3\sqrt{\frac{7}{6}}$$

for all complex numbers with $|z| = 1$.

20. Let $z \in \mathbb{C}^\star$ such that $\left|z^3 + \frac{1}{z^3}\right| \leq 2$. Prove that $\left|z + \frac{1}{z}\right| \leq 2$.

21. Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 1$. Prove that

$$\left|\frac{1}{z} - \frac{1}{2}\right| < \frac{1}{2}$$

22. Show that if $|z| < \frac{1}{2}$, then

$$|(1+i)z^3 + iz| < \frac{3}{4}$$

23. Let $z_1, z_2, z_3 \in \mathbb{C}$ such that $|z_1 z_2 + z_2 z_3 + z_3 z_1| = a$ and $|z_1 z_2 z_3| = b$, where $a, b \neq 0$. Prove that there exists $k \in \{1, 2, 3\}$ for which $|z_k| \leq \frac{3b}{a}$.

24. Let z_1, z_2, z_3 be complex numbers with $|z_1| = |z_2| = |z_3| = R > 0$. Prove that

$$|z_1 - z_2| \cdot |z_2 - z_3| + |z_3 - z_1| \cdot |z_1 - z_2| + |z_2 - z_3| \cdot |z_3 - z_1| \leq 9R^2$$

25. Prove Hlawka's Inequality:

$$|z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1| \leq |z_1| + |z_2| + |z_3| + |z_1 + z_2 + z_3|$$

where z_1, z_2, z_3 are arbitrary complex numbers.

26. *(1962 AMC 12, ♯21)* It is given that one root of $2x^2 + rx + s = 0$ with r and s real numbers, is $3 + 2i$ ($i = \sqrt{-1}$). The value of s is:

(A) undetermined (B) 5 (C) 6 (D) -13 (E) 26

27. *(2007 AMC 12, ♯18)* The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2 + i) = 0$. What is $a + b + c + d$?

(A) 0 (B) 1 (C) 4 (D) 9 (E) 16

28. *(2001 AMC 12, ♯23)* A polynomial of degree 4 with leading coefficient 1 and integer coefficients has two real zeros, both of which are integers. Which of the following can also be a zero of the polynomial?

(A) $\frac{1+i\sqrt{11}}{2}$ (B) $\frac{1+i}{2}$ (C) $\frac{1}{2}+i$ (D) $1+\frac{i}{2}$ (E) $\frac{1+i\sqrt{13}}{2}$

29. *(1982 AMC 12, ♯27)* Suppose $z = a + bi$ is a solution of the polynomial equation:

$$c_4z^4 + ic_3z^3 + c_2z^2 + ic_1z + c_0 = 0$$

where c_0, c_1, c_2, c_3, c_4, a, and b are real constants and $i = \sqrt{-1}$. Which of the following must also be a solution?

(A) $-a - bi$ (B) $a - bi$ (C) $-a + bi$ (D) $b + ai$ (E) none of these

30. *(1972 AMC 12, ♯22)* If $a \pm bi$ ($b \neq 0$, $i = \sqrt{-1}$) are imaginary roots of the equation $x^3 + qx + r = 0$, where a, b, q, and r are real numbers, then q in terms of a and b is:

(A) $a^2 + b^2$ (B) $2a^2 - b^2$ (C) $b^2 - a^2$ (D) $b^2 - 2a^2$ (E) $b^2 - 3a^2$

31. *(1959 AMC 12, ♯27)* Which of the following statements is not true for the equation:

$$ix^2 - x + 2i = 0$$

where $i = \sqrt{-1}$?

(A) the sum of the roots is 2 (B) the discriminant is 9 (C) the roots are imaginary

(D) the roots can be found by using the quadratic formula

(E) the roots can be found by factoring, using imaginary numbers

32. *(2002 AMC 12 A, ♯24)* Find the number of ordered pairs of real numbers (a, b) such that $(a + bi)^{2002} = a - bi$.

(A) 1001 (B) 1002 (C) 2001 (D) 2002 (E) 2004

33. *(2005 AMC 12 B, ♯22)* A sequence of complex numbers z_0, z_1, z_2, ... is defined by the rule:

$$z_{n+1} = \frac{iz_n}{\overline{z_n}}$$

where $\overline{z_n}$ is the complex conjugate of z_n and $i^2 = -1$. Suppose that $|z_0| = 1$ and $z_{2005} = 1$. How many possible values are there for z_0?

(A) 1 (B) 2 (C) 4 (D) 2005 (E) 2^{2005}

34. *(2004 AMC 12 A, ♯23)* A polynomial

$$P(x) = c_{2004}x^{2004} + c_{2003}x^{2003} + \cdots + c_1 x + c_0$$

has real coefficients with $c_{2004} \neq 0$ and 2004 distinct complex zeros $z_k = a_k + b_k i$, $1 \leq k \leq 2004$ with a_k and b_k real, $a_1 = b_1 = 0$, and

$$\sum_{k=1}^{2004} a_k = \sum_{k=1}^{2004} b_k$$

Which of the following quantities can be a nonzero number?

(A) c_0 (B) c_{2003} (C) $b_2 b_3 \ldots b_{2004}$ (D) $\sum_{k=1}^{2004} a_k$ (E) $\sum_{k=1}^{2004} c_k$

35. *(1985 AIME, ♯3)* Find c if a, b, and c are positive integers which satisfy

$$c = (a + bi)^3 - 107i,$$

where $i^2 = -1$.

36. *(2007 AIME I, ♯3)* The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.

37. *(2002 AIME I, ♯12)* Let $F(z) = \dfrac{z+i}{z-i}$ for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that $z_0 = \dfrac{1}{137} + i$ and $z_{2002} = a + bi$, where a and b are real numbers, find $a + b$.