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COMPLEX NUMBERS I - THE ALGEBRAIC FORM

We begin with a review of the main definitions and properties related to complex numbers.

Definitions and terminology: Any complex number z can be uniquely written in the form $z = a + bi$, where a and b are real numbers and $i^2 = -1$.

a is called the real part of z and is denoted by $\text{Re}(z)$, while b is called the imaginary part of z and is denoted by $\text{Im}(z)$. Hence,

$$z = \text{Re}(z) + \text{Im}(z)i$$

The set of all complex numbers is denoted by $\mathbb{C}$.

The following facts are easy to establish:

(a) $z_1 = z_2$ if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

(b) $z \in \mathbb{R}$ if and only if $\text{Im}(z) = 0$.

(c) $z \in \mathbb{C} \setminus \mathbb{R}$ if and only if $\text{Im}(z) \neq 0$.

A complex number of the form $z = bi$, where b is a nonzero real number is called purely imaginary.

Powers of i: We record the following properties of powers of i:

Note that $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 \cdot i = -i$

$i^4 = i^3 \cdot i = 1$, $i^5 = i^4 \cdot i = i$, $i^6 = i^5 \cdot i = -1$, $i^7 = i^6 \cdot i = -i$

One can prove by induction that for any positive integer n,

$$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i$$

Hence $i^n \in \{-1, 1, -i, i\}$ for all integers $n \geq 0$. If n is a negative integer, we have:

$$i^n = (i^{-1})^{-n} = \left(\frac{1}{i}\right)^{-n} = (-i)^{-n}$$

The conjugate of a complex number:

For a complex number $z = a + bi$, the number $\overline{z} = a - bi$ is called the (complex) conjugate of z. The following records the main properties of conjugation:

(1) $z = \overline{z}$ if and only if $z \in \mathbb{R}$.

(2) $z = \overline{\overline{z}}$ for any complex number z.

(3) For any complex number z, the number $z \cdot \overline{z}$ is a nonnegative real number.

(4) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (the conjugate of a sum is the sum of the conjugates)

(5) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ (the conjugate of a product is the product of the conjugates)

(6) $\overline{z^{-1}} = (\overline{z})^{-1}$ for any nonzero complex number z.

(7) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, where $z_2 \neq 0$ (the conjugate of a quotient is the quotient of the conjugates).

(8) $\mathrm{Re}(z) = \frac{z + \overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$ for any complex number z.

REMARKS. Properties (4) and (5) can easily be extended to give:

(4′) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$

(5′) $\overline{z_1 z_2 \ldots z_n} = \overline{z_1} \cdot \overline{z_2} \ldots \overline{z_n}$ for all $z_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$.

As a consequence of (5′) and (6) we have:

(5″) $\overline{z^n} = (\overline{z})^n$ for any integers n and for any $z \in \mathbb{C}$.

The modulus of a complex number:

The number $|z| = \sqrt{a^2 + b^2}$ is called the modulus or the absolute value of the complex number $z = a + bi$.

The following properties are satisfied:

1. $-|z| \leq \mathrm{Re}(z) \leq |z|$ and $-|z| \leq \mathrm{Im}(z) \leq |z|$

2. $|z| \geq 0$ for all $z \in \mathbb{C}$. Moreover, we have $|z| = 0$ if and only if $z = 0$.

3. $|z| = |-z| = |\overline{z}|$

4. $z \cdot \overline{z} = |z|^2$

5. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ (the modulus of a product is the product of the moduli)

6. $|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

7. $|z^{-1}| = |z|^{-1}$, where $z \neq 0$

8. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, where $z_2 \neq 0$ (the modulus of a quotient is the quotient of the moduli)

9. $|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

REMARKS.

(i) The inequality $|z_1+z_2| \leq |z_1|+|z_2|$ becomes an equality if and only if $\mathrm{Re}(z_1\overline{z_2}) = |z_1||z_2|$. This is equivalent to $z_1 = tz_2$, where t is a nonnegative real number.

(ii) The properties (5) and (6) can easily be extended to give:

(5′) $|z_1 \cdot z_2 \ldots z_n| = |z_1| \cdot |z_2| \ldots |z_n|$

(6′) $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$ for all $z_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$.

As a consequence of (5′) and (7) we have $|z^n| = |z|^n$ for any integer n and any complex number z.

1. Calculate:

   (a) $i^{105} + i^{23} + i^{20} - i^{34}$

   (b) $i^{2000} + i^{1999} + i^{201} + i^{82} + i^{47}$

   (c) $E_n = 1 + i + i^2 + i^3 + \cdots + i^n$, for $n \geq 1$

   (d) $i^1 \cdot i^2 \cdot i^3 \ldots i^{2000}$

   (e) $i^{-5} + (-i)^{-7} + (-i)^{13} + i^{-100} + (-i)^{94}$

2. *(1994 AMC 12, ♯12)* Define i such that $i^2 = -1$. What is $(i - i^{-1})^{-1}$?

   (A) 0 (B) $-2i$ (C) $2i$ (D) $-\frac{i}{2}$ (E) $\frac{i}{2}$

3. *(1970 AMC 12, ♯5)* If $f(x) = \dfrac{x^4 + x^2}{x + 1}$, then $f(i)$, where $i = \sqrt{-1}$, is equal to:

(A) $1 + i$ (B) 1 (C) -1 (D) 0 (E) $-1 - i$

4. Consider the polynomial $P(X) = X^{500} + X^{403} + 2X^6 + 3X^5 - 4X^2 + 2X - 6$. Calculate $P(i)$.

5. *(1957 AMC 12, ♯42)* If $S = i^n + i^{-n}$, where $i = \sqrt{-1}$ and n is an integer, then the total number of possible distinct values for S is:

(A) 1 (B) 2 (C) 3 (D) 4 (E) more than 4

6. *(1964 AMC 12, ♯34)* If n is a multiple of 4, the sum $s = 1 + 2i + 3i^2 + \cdots + (n+1)i^n$, where $i = \sqrt{-1}$, equals:

(A) $1 + i$ (B) $\frac{1}{2}(n+2)$ (C) $\frac{1}{2}(n+2-ni)$ (D) $\frac{1}{2}[(n+1)(1-i)+2]$

(E) $\frac{1}{8}(n^2+8-4ni)$

7. *(1965 AMC 12, ♯11)* Consider the statements:

$$I : (\sqrt{-4})(\sqrt{-16}) = \sqrt{(-4)(-16)}$$

$$II : \sqrt{(-4)(-16)} = \sqrt{64}$$

and

$$III : \sqrt{64} = 8$$

Of these the following are incorrect:

(A) none (B) I only (C) II only (D) III only (E) I and III only

8. *(1980 AMC 12, ♯17)* Given that $i^2 = -1$, for how many integers n is $(n+i)^4$ an integer?

(A) none (B) 1 (C) 2 (D) 3 (E) 4

9. Show that $\dfrac{(1+i)^5}{(1-i)^3} = \left(\dfrac{1+i\sqrt{3}}{2}\right)^{60} + \left(\dfrac{1-i\sqrt{3}}{2}\right)^{60}$.

10. Simplify: $z = \dfrac{(1+i)^{200}(6+2i) - (1-i)^{198}(3-i)}{(1+i)^{196}(23-7i) + (1-i)^{194}(10-2i)}$.

11. Simplify the following expression:

$$F = \frac{i - \sqrt{5}}{i + \sqrt{5}} \cdot \frac{2i - \sqrt{5}}{2i + \sqrt{5}} \cdot \frac{3i - \sqrt{5}}{3i + \sqrt{5}} \cdot \frac{4i - \sqrt{5}}{4i + \sqrt{5}}$$

12. Simplify: $A = \left(\frac{4}{\sqrt{3}i - 1} \right)^{12}$.

13. *(1972 AMC 12, ♯3)* If $x = \dfrac{1 - i\sqrt{3}}{2}$, where $i = \sqrt{-1}$, then $\dfrac{1}{x^2 - x}$ is equal to:

(A) -2 (B) -1 (C) $1 + i\sqrt{3}$ (D) 1 (E) 2

14. *(1991 MAΘ)* Find $\left|\dfrac{7 - 24i}{4 + 3i}\right|$.

15. *(2004 MAΘ)* Let $\dfrac{m}{n} = \left|\dfrac{6 + 8i}{-20 + 21i}\right|$, where m and n are positive integers with a greatest common divisor of 1. What does $m + n$ equal?

(A) 34 (B) 39 (C) 941 (D) 1175 (E) None of these

16. *(2004 MAΘ)* If $f(z) = z^2 + 4iz - 4$ and $g(z) = \overline{z}$, then what is the value of $f(g(3+2i))$?

(A) $-21 + 20i$ (B) $-7 - 24i$ (C) $-7 + 24i$ (D) 17 (E) None of these

17. *(2004 MAΘ)* If a and b are natural numbers and $(a+bi)^3$ is a real number, then find $\left|\frac{b}{a}\right|$.

(A) $\frac{\sqrt{3}}{3}$ (B) 1 (C) $\sqrt{3}$ (D) cannot be determined (E) None of these

18. *(2004 MAΘ)* How many complex numbers can be written in the form $a + bi$ where a and b are positive integers and $|a + bi| < 8$?

19. *(1988 AMC 12, ♯21)* The complex number z satisfies $z + |z| = 2 + 8i$. What is $|z|^2$?

(A) 68 (B) 100 (C) 169 (D) 208 (E) 289

20. Let x and y be real numbers, not simultaneously equal to 0. Find the absolute value of the complex number:

$$\varsigma = \frac{x^2 - y^2 + 2xyi}{xy\sqrt{2} + i\sqrt{x^4 + y^4}}$$

21. Show that for any complex number z,

$$\left|z + \frac{1}{2}\right|^2 + i\left|z + \frac{i}{2}\right|^2 - (1+i)|z|^2 - \frac{1}{4}(1+i) = z$$

22. Let a, b, and c be real numbers and $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Compute:

(a) $(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(b) $(a + b)(a + b\omega)(a + b\omega^2)$

(c) $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$

(d) $(a\omega^2 + b\omega)(b\omega^2 + a\omega)$

23. *(1985 AMC 12, ♯23)* If

$$x = \frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad y = \frac{-1 - i\sqrt{3}}{2}$$

where $i^2 = -1$, then which of the following is not correct?

(A) $x^5 + y^5 = -1$ (B) $x^7 + y^7 = -1$ (C) $x^9 + y^9 = -1$ (D) $x^{11} + y^{11} = -1$

(E) $x^{13} + y^{13} = -1$

24. *(1971 AMC 12, ♯22)* If w is one of the imaginary roots of the equation $x^3 = 1$, then the product $(1 - w + w^2)(1 + w - w^2)$ is equal to:

(A) 4 (B) w (C) 2 (D) w^2 (E) 1