MIXED EXERCISES

1. Show that the product of any four consecutive integers plus 1 is a perfect square.

2. Let a, b, c, d be four consecutive terms of an arithmetic progression with ratio r. Show that:
$$abcd + r^4 = (bc - r^2)^2 \quad \text{and} \quad a^4 + b^4 + c^4 + d^4 + 46r^4 = (2bc + 8r^2)^2$$

3. Solve the following equation in $\mathbb{Z} \times \mathbb{Z}$:
$$x(x+1)(x+2)(x+3) = y^2$$

4. Find all real solutions of the equation:
$$x(x+1)(x+2)(x+3) = 24$$

5. Determine the real values of the parameter m such that all roots of the equation:
$$x(x+1)(x+2)(x+3) = m$$
are real.

6. *(2005 USAMO)* Prove that the system
$$\begin{cases} x^6 + x^3 + x^3y + y &= 147^{157} \\ x^3 + x^3y + y^2 + y + z^9 &= 157^{147} \end{cases}$$
has no solutions in integers x, y, and z.

7. The following theorem is known in the literature as *Ptolemy's (First) Theorem:* In a cyclic quadrilateral, the product of the diagonals is equal to the sum of the products of the pairs of opposite sides. Give a proof of Ptolemy's Theorem based on similar triangles.

8. (i) Let $ABCD$ be a convex quadrilateral and let α be the angle formed by the diagonals AC and BD. Prove that the area $[ABCD]$ of quadrilateral $ABCD$ is given by the formula:

$$[ABCD] = \frac{1}{2} \cdot AC \cdot BD \cdot \sin \alpha$$

(ii) Give a proof of Ptolemy's Theorem using part (i) above.

9. Prove *Ptolemy's Second Theorem:* If $ABCD$ is a cyclic quadrilateral, then

$$\frac{AC}{BD} = \frac{AB \cdot AD + BC \cdot CD}{AB \cdot BC + AD \cdot CD}$$