POLYNOMIALS II

1. Consider the polynomial

$$P(x) = (x-2)(2x-3)^2(3x-4)^3 \dots (2009x-2010)^{2009}$$

When $P(x)$ is written in standard form, what is the sum of its coefficients?

2. *(1966 AMC 12, ♯36)* Let $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$ be an identity in x. If we let $s = a_0 + a_2 + a_4 + \cdots + a_{2n}$, then s equals:

(A) 2^n (B) $2^n + 1$ (C) $\dfrac{3^n - 1}{2}$ (D) $\dfrac{3^n}{2}$ (E) $\dfrac{3^n + 1}{2}$

3. *(1961 AMC 12, ♯5)* Let $S = (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1$. Then S equals:

(A) $(x-2)^4$ (B) $(x-1)^4$ (C) x^4 (D) $(x+1)^4$ (E) x^4+1

4. Show that the polynomial $f = (x-1)^{12n+1} + x^{12n+1} - 2x + 1$ is divisible by $x(x-1)\left(x - \frac{1}{2}\right)$.

5. *(1970 AMC 12, ♯11)* If two factors of $2x^3 - hx + k$ are $x+2$ and $x-1$, the value of $|2h - 3k|$ is:

(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

6. *(1961 AMC 12, ♯22)* If $3x^3 - 9x^2 + kx - 12$ is divisible by $x - 3$, then it is also divisible by:

   (A) $3x^2 - x + 4$ (B) $3x^3 - 4$ (C) $3x^2 + 4$ (D) $3x - 4$ (E) $3x + 4$

7. *(1965 AMC 12, ♯19)* If $x^4 + 4x^3 + 6px^2 + 4qx + r$ is exactly divisible by $x^3 + 3x^2 + 9x + 3$, the value of $(p + q)r$ is:

   (A) -18 (B) 12 (C) 15 (D) 27 (E) 45

8. *(1965 AMC 12, ♯27)* When $y^2 + my + 2$ is divided by $y - 1$ the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $y + 1$ the quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$, then m is:

(A) 0 (B) 1 (C) 2 (D) -1 (E) an undetermined constant

9. *(1969 AMC 12, ♯34)* The remainder R obtained by dividing x^{100} by $x^2 - 3x + 2$ is a polynomial of degree less than 2. Then R may be written as:

(A) $2^{100} - 1$ (B) $2^{100}(x - 1) - (x - 2)$ (C) $2^{100}(x - 3)$ (D) $x(2^{100} - 1) + 2(2^{99} - 1)$
(E) $2^{100}(x + 1) - (x + 2)$

10. Show that $x^{24} - x^{13} - x^7 + x^6 + 1 \vdots x^4 - x^3 + x^2 - x + 1$.

11. Let $P(x) = x^{1985} + x^{1984} + \cdots + x^2 + x + 1$ and $Q(x) = x^{661} + x^{660} + \cdots + x^2 + x + 1$. Show that $P(x)$ is divisible by $Q(x)$.

12. *(Polish Mathematical Olympiad)* Prove that the polynomial $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is divisible by the polynomial $x^4 + x^3 + x^2 + x + 1$.

13. Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

14. Find the remainder when $x^{60} - 1$ is divided by $x^3 - 2$.

15. *(1963 Putnam)* For what integers a does $x^2 - x + a$ divide $x^{13} + x + 90$?

16. *(1999 AMC 12, ♯17)* Let $P(x)$ be a polynomial such that when $P(x)$ is divided by $x - 19$, the remainder is 99, and when $P(x)$ is divided by $x - 99$, the remainder is 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

(A) $-x + 80$ (B) $x + 80$ (C) $-x + 118$ (D) $x + 118$ (E) 0

17. If $P(a) = a$, $P(b) = b$, and $P(c) = c$ where a, b, and c are distinct numbers, what is the remainder when the polynomial P is divided by $(x - a)(x - b)(x - c)$?

18. *(1962 AMC 12, ♯9)* When $x^9 - x$ is factored as completely as possible into polynomials and monomials with integral coefficients, the number of factors is:

(A) more than 5 (B) 5 (C) 4 (D) 3 (E) 2

19. *(Polish Mathematical Olympiad)* Factor $(a + b + c)^3 - (a^3 + b^3 + c^3)$.

20. *(Polish Mathematical Olympiad)* Factor $(a+b)^7 - a^7 - b^7$.

21. *(Polish Mathematical Olympiad)* Factor $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$.

22. *(1964 AMC 12, ♯25)* The set of values of m for which $x^2 + 3xy + x + my - m$ has two factors, with integer coefficients, which are linear in x and y, is precisely:

(A) 0, 12, −12 (B) 0, 12 (C) 12, −12 (D) 12 (E) 0

23. *(1963 AMC 12, ♯21)* The expression $x^2 - y^2 - z^2 + 2yz + x + y - z$ has:

(A) no linear factor with integer coefficients and integer exponents

(B) the factor $-x + y + z$ (C) the factor $x - y - z + 1$ (D) the factor $x + y - z + 1$

(E) the factor $x - y + z + 1$

24. *(1962 AMC 12, ♯34)* For what real values of K does $x = K^2(x-1)(x-2)$ have real roots?

(A) none (B) $-2 < K < 1$ (C) $-2\sqrt{2} < K < 2\sqrt{2}$ (D) $K > 1$ (E) all

25. *(1962 AMC 12, ♯21)* It is given that one root of $2x^2 + rx + s = 0$, with r and s real numbers, is $3 + 2i$ $(i = \sqrt{-1})$. The value of s is:

(A) undetermined (B) 5 (C) 6 (D) -13 (E) 26

26. *(1972 AMC 12, ♯22)* If $a \pm bi$ $(b \neq 0,\ i = \sqrt{-1})$ are imaginary roots of the equation $x^3 + qx + r = 0$, where a, b, q, and r are real numbers, then q in terms of a and b is:

(A) $a^2 + b^2$ (B) $2a^2 - b^2$ (C) $b^2 - a^2$ (D) $b^2 - 2a^2$ (E) $b^2 - 3a^2$

27. *(1963 AMC 12, ♯24)* Consider equations of the form $x^2 + bx + c = 0$. How many such equations have real roots and have coefficients b and c selected from the set of integers $\{1, 2, 3, 4, 5, 6\}$?

(A) 20 (B) 19 (C) 18 (D) 17 (E) 16

28. *(1964 AMC 12, ♯7)* Let n be the number of real values of p for which the roots of

$$x^2 - px + p = 0$$

are equal. Then n equals:

(A) 0 (B) 1 (C) 2 (D) a finite number greater than 2

(E) an infinitely large number

29. *(1964 AMC 12, ♯8)* The smaller root of the equation

$$\left(x-\frac{3}{4}\right)\left(x-\frac{3}{4}\right)-\left(x-\frac{3}{4}\right)\left(x-\frac{1}{2}\right)=0$$

is:

(A) $-\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{5}{8}$ (D) $\frac{3}{4}$ (E) 1

30. *(1970 AMC 12, ♯14)* Consider $x^2+px+q=0$ where p and q are positive numbers. If the roots of this equation differ by 1, then p equals:

(A) $\sqrt{4q+1}$ (B) $q-1$ 9C) $-\sqrt{4q+1}$ (D) $q+1$ (E) $\sqrt{4q-1}$

31. *(1964 AMC 12, ♯30)* The larger root minus the smaller root of the equation:

$$(7+4\sqrt{3})x^2+(2+\sqrt{3})x-2=0$$

is:

(A) $-2+3\sqrt{3}$ (B) $2-\sqrt{3}$ (C) $6+3\sqrt{3}$ (D) $6-3\sqrt{3}$ (E) $3\sqrt{3}+2$

32. *(1966 AMC 12, ♯30)* If three of the roots of $x^4+ax^2+bx+c=0$ are 1, 2, and 3, then the value of $a+c$ is:

(A) 35 (B) 24 (C) -12 (D) -61 (E) -63

33. *(1977 USAMO, ♯1)* Determine all pairs of positive integers (m, n) such that

$$(1 + x^n + x^{2n} + \cdots + x^{mn}) \quad \text{is divisible by} \quad (1 + x + x^2 + \cdots + x^m)$$

34. *(1984 USAMO, ♯1)* The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.

35. *(1977 USAMO, ♯3)* If a and b are two of the roots of $x^4 + x^3 - 1 = 0$, prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

36. *(Polish Mathematical Olympiad)* Prove that if

$$\frac{a-b}{1+ab}+\frac{b-c}{1+bc}+\frac{c-a}{1+ca}=0$$

then at least two of the numbers a, b, c are equal.

37. *(Polish Mathematical Olympiad)* Prove that if

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{ab+bc+ca}$$

then two of the numbers a, b, and c are opposite numbers.

38. Does there exist a polynomial P with integer coefficients such that $P(7) = 5$ and $P(3) = 4$?

39. *(1986 BMO, ♯4)* Let $P(x)$ be a polynomial with integer coefficients such that

$$P(21) = 17, \quad P(32) = -247, \quad P(37) = 33$$

Prove that if $P(N) = N + 51$ for some integer N, then $N = 26$.