POLYNOMIALS I

1. *(1983 AMC 12, ♯18)* Let f be a polynomial function such that for all real x:

$$f(x^2+1) = x^4 + 5x^2 + 3$$

For all real x, $f(x^2-1)$ is:

(A) $x^4 + 5x^2 + 1$ (B) $x^4 + x^2 - 3$ (C) $x^4 - 5x^2 + 1$ (D) $x^4 + x^2 + 3$

(E) none of these

2. Find all polynomials $P(x)$ with real coefficients satisfying the following property:

$$P(x+1) = P(x) + 3x^2 + 3x + 1$$

for all real numbers x.

3. *(1987 AMC 12, ♯24)* How many polynomial functions f of degree ≥ 1 satisfy:

$$f(x^2) = [f(x)]^2 = f(f(x)) \text{ ?}$$

(A) 0 (B) 1 (C) 2 (D) finitely many but more than 2 (E) infinitely many

4. *(Canada 1975)* Let k be a positive integer. Find all polynomials with real coefficients which satisfy the equation:

$$P(P(x)) = [P(x)]^k$$

5. *(2004 Mu Alpha Theta National Convention)* Consider the polynomial $f(x)$ with degree n and integer coefficients. Given that

$$f(1) = f(2) = f(3) = f(4) = 2004,$$

prove that there is no integer m such that $f(m) = 1$.

6. Let $P(x)$ be a polynomial with integer coefficients and let a and b be relatively prime positive integers such that both $P(a)$ and $P(b)$ are divisible by ab. Show that $P(a+b)$ is divisible by ab.

7. Find the unique polynomial P of degree n that satisfies

$$P(j) = \frac{1}{j+1} \quad \text{for} \quad j = 0, 1, 2, \ldots, n$$

8. *(1975 USAMO, ♯3)* If $P(x)$ denotes a polynomial of degree n such that

$$P(k) = \frac{k}{k+1} \quad \text{for} \quad k = 0, 1, 2, \ldots, n,$$

determine $P(n+1)$.

9. Find all polynomials $P(x)$ with real coefficients and degree n satisfying:

$$P(1) + P(x) + P(x^2) + \cdots + P(x^n) = (1 + x + x^2 + \cdots + x^n)P(x)$$

for all real numbers x.

10. *(1976 USAMO, ♯5)* If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.