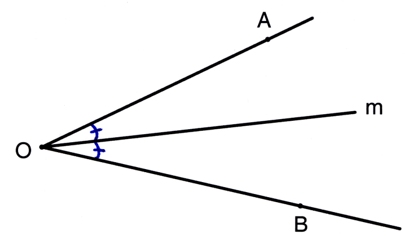
# ANGLE BISECTORS

Recall that the *bisector* of an angle is the ray that divides the angle into two congruent angles. The most important results about angle bisectors can be found below:

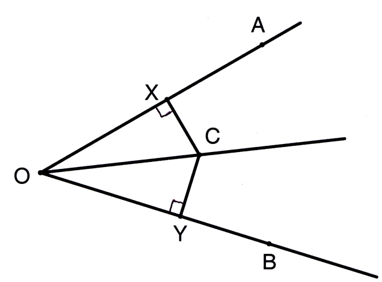



*Ray m is the angle bisector of ∠AOB*

*FACT 1.* The bisector of an angle consists of all points that are equidistant from the sides of the angle.

In other words: Let $C$ be a point interior to the angle $\measuredangle AOB$. Let $X$ and $Y$ be on $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively, such that $\overline{CX} \perp \overrightarrow{OA}$ and $\overline{CY} \perp \overrightarrow{OB}$. Then:

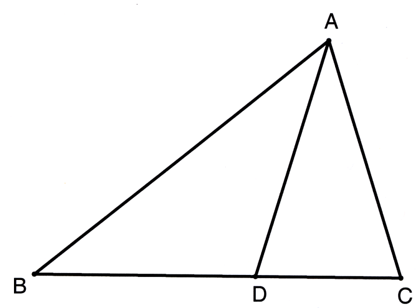
$$C \text{ is on the bisector of } \measuredangle AOB \text{ if and only if } \overline{CX} \cong \overline{CY}.$$




*Proof.*

*FACT 2. (The Angle Bisector Theorem)* In any triangle, an angle bisector divides the opposite side into segments proportional to the sides of the angle. That is, if $ABC$ is a triangle and $\overline{AD}$ is the bisector of $\measuredangle A$, then

$$\frac{BD}{DC} = \frac{AB}{AC}$$

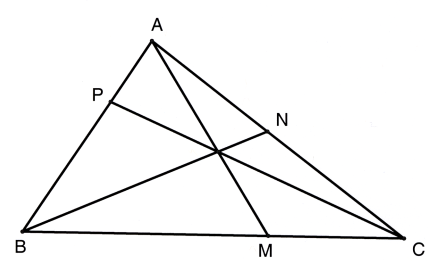



*Proof.*

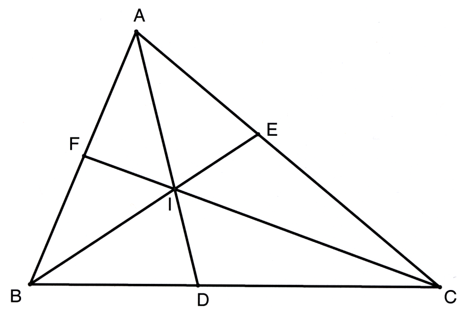
Recall now *Ceva's Theorem*:

*Ceva's Theorem.* Let $M$, $N$, and $P$ be points on the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ of triangle $ABC$, respectively. Then:

$$AM,\ BN,\ \text{and}\ CP \text{ are concurrent} \Leftrightarrow \frac{BM}{MC} \cdot \frac{CN}{NA} \cdot \frac{AP}{PB} = 1$$



*FACT 3.* In any triangle, the angle bisectors are concurrent. That is, if $AD$, $BE$, and $CF$ are the bisectors of $\measuredangle A$, $\measuredangle B$, and $\measuredangle C$, respectively, then $AD$, $BE$, and $CF$ are concurrent.



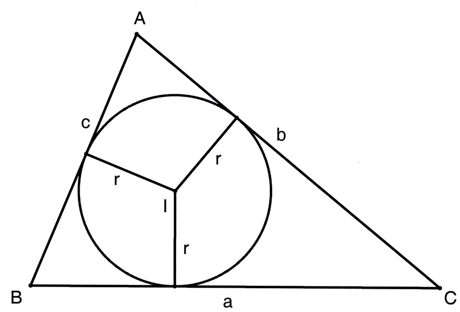

*Proof 1.*

*Proof 2.*

Let $ABC$ be a triangle. The intersection of the angle bisectors of the triangle is usually denoted by $I$ and is called the *incenter* of the triangle. Note that $I$ is equidistant from the sides of the triangle. This common distance from the incenter to the sides is called the *inradius*, because the circle with center $I$ and this radius is tangent to all three sides of the triangle. The inradius is usually denoted by $r$.

*FACT 4.* If $ABC$ is a triangle with area $S$, inradius $r$, and semiperimeter $p$, then

$$r = \frac{S}{p}$$

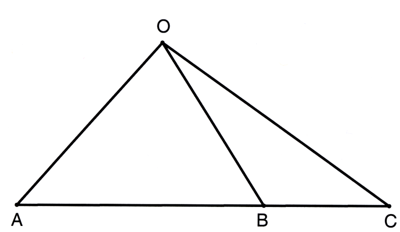



*Proof.*

Recall *Stewart's Theorem*, named after the eighteenth-century Scottish mathematician Matthew Stewart, although forms of the theorem were known as long ago as the fourth century A.D.

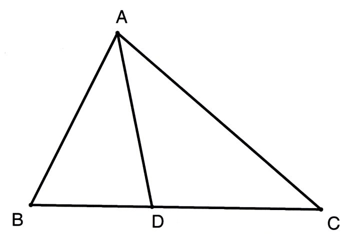
*Stewart's Theorem.* Let $A$, $B$, and $C$ be collinear points, in this order, and let $O$ be a point not on the line determined by $A$, $B$, and $C$. Then:

$$OA^2 \cdot BC + OC^2 \cdot AB - OB^2 \cdot AC = AB \cdot BC \cdot AC$$



*FACT 5.* Let $\overline{AD}$ be the bisector of the angle $\measuredangle BAC$ of the triangle $ABC$. Then:

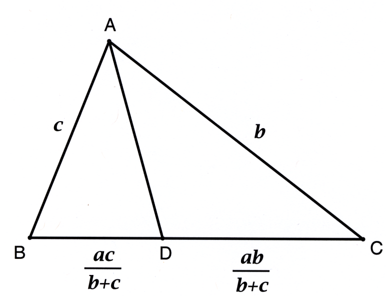
$$AD^2 = AB \cdot AC - BD \cdot DC$$




*Proof.*

*FACT 6.* Let $\overline{AD}$ be the bisector of the angle $\measuredangle BAC$ of the triangle $ABC$. Let $BC = a$, $AC = b$, and $AB = c$. Then

$$BD = \frac{ac}{b+c} \quad \text{and} \quad DC = \frac{ab}{b+c}$$




*Proof.*

Similarly, if $\overline{BE}$ is the bisector of $\measuredangle ABC$, we have $CE = \dfrac{ab}{a+c}$ and $AE = \dfrac{bc}{a+c}$.
If $\overline{CF}$ is the bisector of $\measuredangle BCA$, then $AF = \dfrac{bc}{a+b}$ and $FB = \dfrac{ac}{a+b}$.

For a triangle $ABC$ with side lengths $BC = a$, $AC = b$, $AB = c$, and semiperimeter $p$, the lengths of the angle bisectors $AD$, $BE$, and $CF$ are usually denoted by $\ell_a$, $\ell_b$, and $\ell_c$. Using Facts 5 and 6 we can deduce the following result:

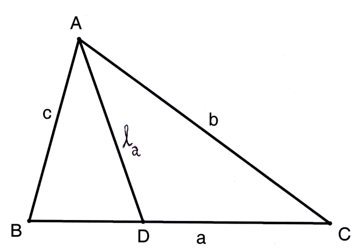
*FACT 7.* $\ell_a = \dfrac{2}{b+c}\sqrt{bcp(p-a)}$, $\ell_b = \dfrac{2}{a+c}\sqrt{acp(p-b)}$, and $\ell_c = \dfrac{2}{a+b}\sqrt{abp(p-c)}$.

*Proof.*

Recall that the area $S$ of the triangle can be calculated by the formulas:

$$S = \frac{ab\sin C}{2} = \frac{bc\sin A}{2} = \frac{ac\sin B}{2}$$

*FACT 8.* $\ell_a = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$, $\ell_b = \dfrac{2ac}{a+c}\cos\dfrac{B}{2}$, and $\ell_c = \dfrac{2ab}{a+b}\cos\dfrac{C}{2}$.




*Proof.*

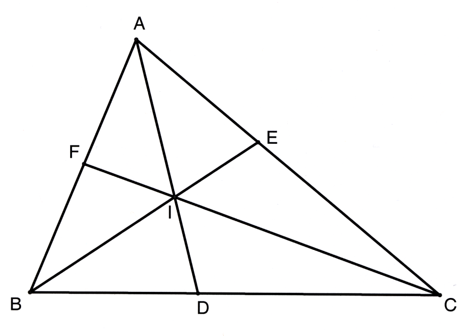
As a corollary of our previous two results we obtain formulas for the cosines of the half angles of $\triangle ABC$ in terms of the sides:

*FACT 9.* $\cos\dfrac{A}{2} = \sqrt{\dfrac{p(p-a)}{bc}}$, $\cos\dfrac{B}{2} = \sqrt{\dfrac{p(p-b)}{ac}}$, and $\cos\dfrac{C}{2} = \sqrt{\dfrac{p(p-c)}{ab}}$.

*Proof.*

*FACT 10.* Let $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ be the angle bisectors of $\triangle ABC$. Let $I$ be the incenter. Then:

$$\frac{AI}{ID} = \frac{b+c}{a}, \qquad \frac{BI}{IE} = \frac{a+c}{b}, \qquad \frac{CI}{IF} = \frac{a+b}{c}$$




*Proof.*

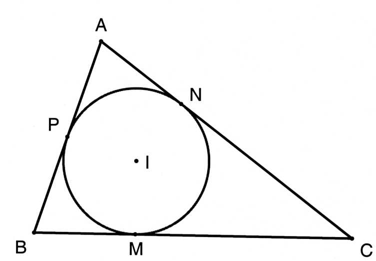
*FACT 11.* If $I$ is the incenter of $\triangle ABC$, then

$$AI = \sqrt{\frac{bc(p-a)}{p}}, \quad BI = \sqrt{\frac{ac(p-b)}{p}}, \quad CI = \sqrt{\frac{ab(p-c)}{p}}$$

*Proof.*

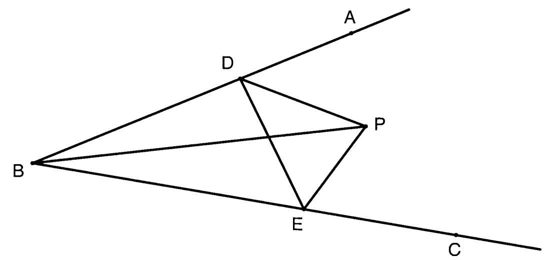
*FACT 12.* Let $M$, $N$, and $P$ be the points of contact of the incenter with the sides of the triangle $ABC$, where $M$ is on $\overline{BC}$, $N$ is on $\overline{AC}$, and $P$ is on $\overline{AB}$. Then:

$$AN = AP = p - a, \quad BM = BP = p - b, \quad CM = CN = p - c$$




*Proof.*

1. In the picture below, $\overrightarrow{DP}$ bisects $\measuredangle ADE$ and $\overrightarrow{EP}$ bisects $\measuredangle DEC$. Prove that $\overrightarrow{BP}$ bisects $\measuredangle ABC$.




2. *(Timisoara Mathematics Review, 1987 Middle School Annual Contest)* Let $ABC$ be a right triangle and let $D$ be a point on the hypotenuse $\overline{BC}$. Let $\overline{DE}$ and $\overline{DF}$ be the bisectors of the angles $\measuredangle ADB$, and $\measuredangle ADC$, respectively, where $E$ is on $\overline{AB}$ and $F$ is on $\overline{AC}$. Show that:

$$AB \cdot AE + AC \cdot AF = AD \cdot BC$$

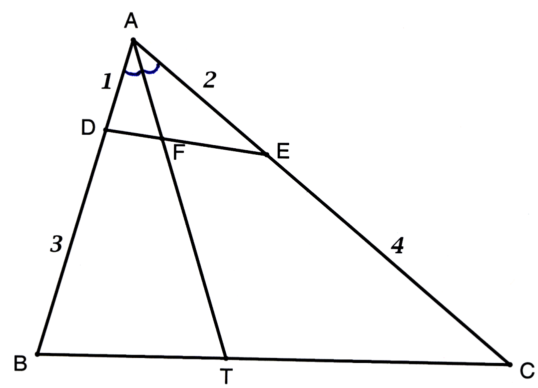
3. Prove *Steiner-Lehmus Theorem:* Let $ABC$ be a triangle. Then $\triangle ABC$ is isosceles if and only if it has two congruent angle bisectors.

4. *(1989 Romanian Math Olympiad, School Competition, 10th grade)* Show that the triangle $ABC$ is isosceles if and only if

$$c(a+b)\cos\frac{B}{2} = b(a+c)\cos\frac{C}{2}$$

5. *(1989 ARML, Team Question ♯4)* In triangle $ABC$, angle bisectors $\overline{AD}$ and $\overline{BE}$ intersect at $P$. If $BC = 3$, $AC = 5$, $AB = 7$, $BP = x$, and $PE = y$, compute the ratio $x : y$, where $x$ and $y$ are relatively prime integers.

6. *(1992 ARML, Individual Question ♯18)* In triangle $ABC$, points $D$ and $E$ are on $\overline{AB}$ and $\overline{AC}$, and angle-bisector $\overline{AT}$ intersects $\overline{DE}$ at $F$ (as shown in the diagram). If $AD = 1$, $BD = 3$, $AE = 2$, and $EC = 4$, compute the ratio $AF : AT$.



7. Prove that the inradius of a right triangle with legs of lengths $a$ and $b$ and hypotenuse $c$ is $(a+b-c)/2$.

8. The sides of a right triangle $ABC$ all have integer lengths. Prove that the inradius of $\triangle ABC$ also has integer length.

9. *(1993 AIME, ♯15)* Let $\overline{CH}$ be an altitude of $\triangle ABC$. Let $R$ and $S$ be the points where the circles inscribed in triangles $ACH$ and $BCH$ are tangent to $\overline{CH}$. If $AB = 1995$, $AC = 1994$, and $BC = 1993$, then $RS$ can be expressed as $m/n$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

10. *(2008 AMC 12A, ♯20)* Triangle $ABC$ has $AC = 3$, $BC = 4$, and $AB = 5$. Point $D$ is on $\overline{AB}$ and $\overline{CD}$ bisects the right angle. The inscribed circles of $\triangle ADC$ and $\triangle BCD$ have radii $r_a$ and $r_b$, respectively. What is $\frac{r_a}{r_b}$?

(A) $\frac{1}{28}(10-\sqrt{2})$ (B) $\frac{3}{56}(10-\sqrt{2})$ (C) $\frac{1}{14}(10-\sqrt{2})$ (D) $\frac{5}{56}(10-\sqrt{2})$
(E) $\frac{3}{28}(10-\sqrt{2})$

11. *(1997 ARML, Team Contest ♯6)* In $\triangle ABC$, $D$ lies on $\overline{AC}$ so that $\measuredangle ABD \cong \measuredangle DBC = \theta$. If $AB = 4$, $BC = 16$, and $BD = \frac{2}{\cos\theta}$, then $BD = \frac{a}{\sqrt{b}}$ for relatively prime integers $a$ and $b$. Compute the ordered pair $(a, b)$.

12. *(2001 AIME I, ♯4)* In triangle $ABC$, angles $A$ and $B$ measure 60 degrees and 45 degrees, respectively. The bisector of angle $A$ intersects $BC$ at $T$, and $AT = 24$. The area of the triangle $ABC$ can be written in the form $a + b\sqrt{c}$, where $a$, $b$, and $c$ are positive integers, and $c$ is not divisible by the square of any prime. Find $a + b + c$.

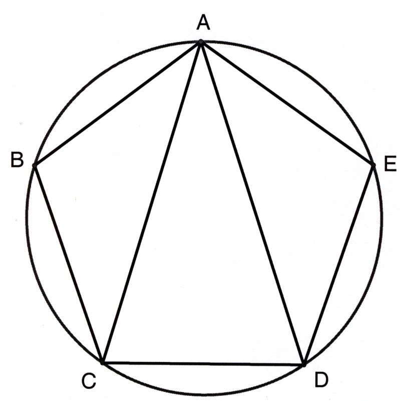
13. *(2001 AIME I, ♯7)* Triangle $ABC$ has $AB = 21$, $AC = 22$, and $BC = 20$. Points $D$ and $E$ are located on $\overline{AB}$ and $\overline{AC}$, respectively, such that $DE$ is parallel to $\overline{BC}$ and contains the center of the inscribed circle of triangle $ABC$. Then $DE = m/n$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

14. *(2001 AIME II, ♯7)* Let $\triangle PQR$ be a right triangle with $PQ = 90$, $PR = 120$, and $QR = 150$. Let $\mathfrak{C}_1$ be the inscribed circle. Construct $\overline{ST}$, with $S$ on $\overline{PR}$ and $T$ on $\overline{QR}$ such that $\overline{ST}$ is perpendicular to $\overline{QR}$ and tangent to $\mathfrak{C}_1$. Construct $\overline{UV}$ with $U$ on $\overline{PQ}$ and $V$ on $\overline{QR}$ such that $\overline{UV}$ is perpendicular to $\overline{PQ}$ and tangent to $\mathfrak{C}_1$. Let $\mathfrak{C}_2$ be the inscribed circle of $\triangle RST$ and $\mathfrak{C}_3$ the inscribed circle of $\triangle QUV$. The distance between the centers of $\mathfrak{C}_2$ and $\mathfrak{C}_3$ can be written as $\sqrt{10n}$. What is $n$?

15. In $\triangle ABC$, $\overline{AB} \cong \overline{AC}$, $\angle BAC = 36°$, and $BC = 1$. Find $AB$.

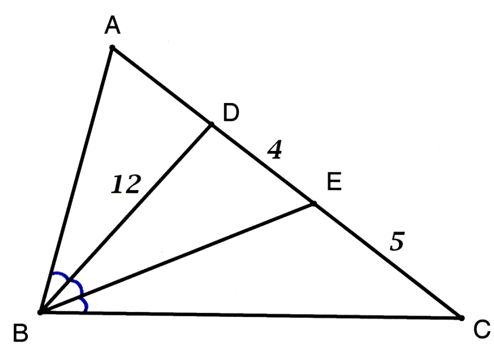
16. Let $ABCDE$ be a regular pentagon inscribed in a unit circle. Prove that the product of the lengths of the two sides and the diagonals from a vertex is 5, that is, in the figure below, prove that

$$AB \cdot AC \cdot AD \cdot AE = 5$$



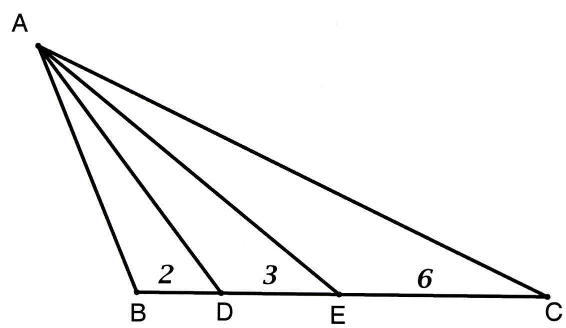
17. *(Timisoara Mathematics Review, 1987 Middle School Annual Contest)* Show that there does not exist a triangle whose incircle divides an angle bisector into three congruent segments.

18. In triangle $ABC$, rays $\overrightarrow{BD}$ and $\overrightarrow{BE}$ trisect $\angle ABC$. If $DE = 4$, $EC = 5$, and $BD = 12$, find $BC$, $BE$, and $AD$.



19. Is it possible for the angle trisectors of angle $\measuredangle A$ of triangle $ABC$ to divide side $BC$ into three congruent segments? Justify your answer.

20. *(1981 AMC 12, ♯25)* In triangle $ABC$ in the adjoining figure, $AD$ and $AE$ trisect $\sphericalangle BAC$. The lengths of $BD$, $DE$, and $EC$ are 2, 3, and 6, respectively. The length of the shortest side of $\triangle ABC$ is:




(A) $2\sqrt{10}$ (B) 11 (C) $6\sqrt{6}$ (D) 6 (E) not uniquely determined by the given information

21. Let $\ell_1$ and $\ell_2$ be the trisectors of $\measuredangle BAC$ of triangle $ABC$. Show that:

$$\ell_1 \cdot \left(\frac{1}{c} + \frac{1}{\ell_2}\right) = \ell_2 \cdot \left(\frac{1}{b} + \frac{1}{\ell_1}\right)$$

22. Let $\overline{AD}$ and $\overline{AE}$ be the trisectors of $\measuredangle BAC$ of triangle $ABC$, where $D$ and $E$ are on $\overline{BC}$. Show that:

$$\frac{1}{AD^2} - \frac{1}{AE^2} = \frac{2R\sin\dfrac{A}{3}}{BC}\left(\frac{1}{AB^2} - \frac{1}{AC^2}\right)$$