

INEQUALITIES II

1. Prove the *Arithmetic-Geometric Mean Inequality* geometrically. That is, show that if x and y are two positive real numbers, then

$$x + y \geq 2\sqrt{xy}$$

2. Using the previous exercise, or otherwise, prove *Padoa's Inequality*: If a , b , and c are the sides of a triangle, then:

$$abc \geq (a + b - c)(b + c - a)(c + a - b)$$

Alessandro Padoa (1868 - 1937) was an Italian mathematician and logician.

3. Let a , b , and c be the sides of a triangle ABC and let S be its area. Prove that:

$$a + b + c = \frac{16S^2}{abc} \text{ if and only if } \triangle ABC \text{ is equilateral.}$$

4. Prove *Euler's Inequality*: If R and r are the circumradius and the inradius of a triangle, respectively, then:

$$R \geq 2r$$

The next result is one of the most useful inequalities; proved first by Augustin Cauchy in 1821, and extended to integrals by Viktor Yakovlevich Bunyakovski in 1859, the inequality has been rediscovered by Hermann Amandus Schwarz in 1888, and is widely used in linear algebra, analysis, and probability theory.

Cauchy-Bunyakovsky-Schwarz Inequality:

Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be any nonzero real numbers. Then:

$$(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$$

and equality holds if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$$

We will present below three different proofs:

Proof 1.

Proof 2. This is based on the *The Rearrangement Inequality*:

Let $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$ be real numbers. Let $\{c_1, c_2, \dots, c_n\}$ be any permutation of b_1, b_2, \dots, b_n . The *Rearrangement Inequality* states that:

$$a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1 \leq a_1 c_1 + a_2 c_2 + \dots + a_n c_n \leq a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

For the first inequality, equality happens when $c_i = b_{n-i+1}$ for $1 \leq i \leq n$, with the convention that $c_i \geq c_{i+1}$ whenever $a_i = a_{i+1}$. In the second case, equality happens if and only if $c_i = b_i$, but only with the convention that $c_i \leq c_{i+1}$ whenever $a_i = a_{i+1}$.

Proof 3. This is based on the following very useful lemma:

LEMMA. If $a, b, x,$ and y are real numbers and $x, y > 0$, then the following inequality holds:

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$$

5. Prove that if a_1, a_2, \dots, a_n are positive real numbers, then:

$$(a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \geq n^2$$

6. Prove that for all real numbers a_1, a_2, \dots, a_n , the following inequality holds:

$$n(a_1^2 + a_2^2 + \dots + a_n^2) \geq (a_1 + a_2 + \dots + a_n)^2$$

7. If $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are positive real numbers, prove that:

$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \dots + \sqrt{a_n b_n} \leq \sqrt{a_1 + a_2 + \dots + a_n} \sqrt{b_1 + b_2 + \dots + b_n}$$

8. Prove *Minkowski's Inequality*: If $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are positive real numbers, then:

$$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + \dots + (a_n + b_n)^2} \leq \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} + \sqrt{b_1^2 + b_2^2 + \dots + b_n^2}$$

9. If $\sigma(1), \sigma(2), \dots, \sigma(n)$ is a permutation of the set of the first n positive integers, prove that:

$$\frac{\sigma(1)}{1^2} + \frac{\sigma(2)}{2^2} + \dots + \frac{\sigma(n)}{n^2} \geq 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

10. (*Mathematics and Informatics Quarterly, 1991*) Find the minimum and maximum values of the function:

$$f(x, y) = 2 \sin x \cos y + 3 \sin x \sin y + 6 \cos x$$

11. Let a and b be real numbers such that $a + b = 2$. Prove that $a^4 + b^4 \geq 2$.

12. Let a , b , and c be positive real numbers. Prove that:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

13. (*Matematika, Sofia, 1978*) Let x_1, x_2, x_3, x_4, x_5 be positive real numbers. Prove that:

$$\frac{x_1}{x_2+x_3} + \frac{x_2}{x_3+x_4} + \frac{x_3}{x_4+x_5} + \frac{x_4}{x_5+x_1} + \frac{x_5}{x_1+x_2} \geq \frac{5}{2}$$