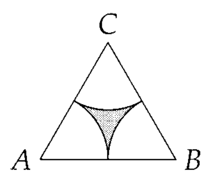
San Jose Math Circle February 14, 2009

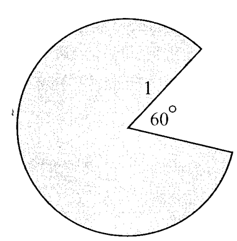
CIRCLES AND FUNKY AREAS - PART II

Warm-up Exercises

1. In the diagram below, $\triangle ABC$ is equilateral with side length 6. Arcs are drawn centered at the vertices connecting midpoints of consecutive sides, as shown. Find the area of the shaded region.

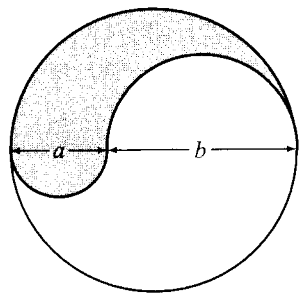


2. In an arcade game, the "monster" is the sector of radius 1 cm, as shown in the figure. The missing piece (the mouth) has central angle 60°. What is the perimeter of the monster in cm?



(A) $\pi + 2$ (B) 2π (C) $\dfrac{5}{3}\pi$ (D) $\dfrac{5}{6}\pi + 2$ (E) $\dfrac{5}{3}\pi + 2$

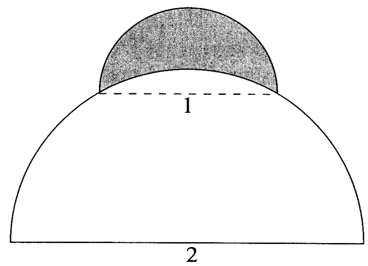
3. *(1998 AMC 12, ♯16)* The figure shown is the union of a circle and semicircles of diameters a and b, all of whose centers are collinear. What is the ratio of the area of the shaded region to that of the unshaded region?



(A) $\sqrt{\frac{a}{b}}$ (B) $\frac{a}{b}$ (C) $\frac{a^2}{b^2}$ (D) $\frac{a+b}{2b}$ (E) $\frac{a^2+2ab}{b^2+2ab}$

4. *(2004 AIME 2, ♯1)* A chord of a circle is perpendicular to a radius at the midpoint of the radius. The ratio of the area of the larger of the two regions into which the chord divides the circle to the smaller can be expressed in the form $\frac{a\pi + b\sqrt{c}}{d\pi - e\sqrt{f}}$, where a, b, c, d, e, and f are positive integers, a and e are relatively prime, and neither c nor f is divisible by the square of any prime. Find the remainder when the product $a \cdot b \cdot c \cdot d \cdot e \cdot f$ is divided by 1000.

5. A semicircle of diameter 1 sits on top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a *lune*. What is the area of this lune?



(A) $\frac{1}{12}(2\pi - 3\sqrt{3})$ (B) $\frac{1}{12}(3\sqrt{3} - \pi)$ (C) $\frac{1}{24}(6\sqrt{3} - \pi)$ (D) $\frac{1}{24}(6\sqrt{3} + \pi)$
(E) $\frac{1}{12}(3\sqrt{3} + \pi)$

DEFINITIONS.

(a) *Tangent circles* are circles that intersect each other at exactly one point.

(b) Two circles are called *externally tangent* if each of the tangent circles lies outside the other.

(c) Two circles are called *internally tangent* if one of the tangent circles lies inside the other.

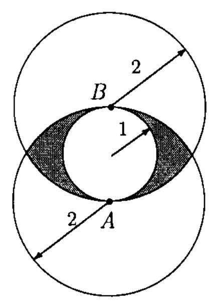
THEOREM.

(a) Two tangent circles have a common tangent line.

(b) The centers of two tangent circles and their tangency point are collinear.

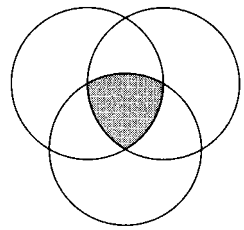
6. Circle $\odot A$ passes through B and circle $\odot B$ passes through A. Given that $AB = 6$, find the area of the shaded region common to both circles.

7. *(2004 AMC 10A, ♯25)* A circle of radius 1 is internally tangent to two circles of radius 2 at points A and B, where AB is a diameter of the smaller circle. What is the area of the region, shaded in the figure, that is outside the smaller circle and inside each of the two larger circles?



(A) $\frac{5}{3}\pi - 3\sqrt{2}$ (B) $\frac{5}{3}\pi - 2\sqrt{3}$ (C) $\frac{8}{3}\pi - 3\sqrt{3}$ (D) $\frac{8}{3}\pi - 3\sqrt{2}$ (E) $\frac{8}{3}\pi - 2\sqrt{3}$

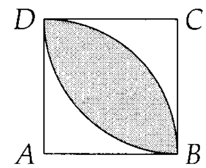
8. Three circles of radius 12 lie in a plane such that each passes through the center of the other two. Find the area common to all three circles.



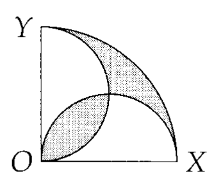
9. *(2009 AMC 10A, ♯19)* Circle A has radius 100. Circle B has an integer radius $r < 100$ and remains internally tangent to circle A as it rolls once around the circumference of circle A. The two circles have the same points of tangency at the beginning and end of circle B's trip. How many possible values can r have?

(A) 4 (B) 8 (C) 9 (D) 50 (E) 90

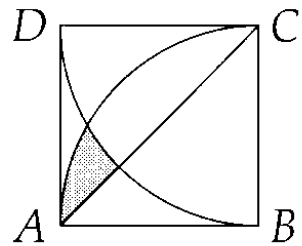
10. In the diagram below, quarter-circles have been drawn centered at vertices A and C of square $ABCD$. Given that $AB = 6$, find the shaded area.



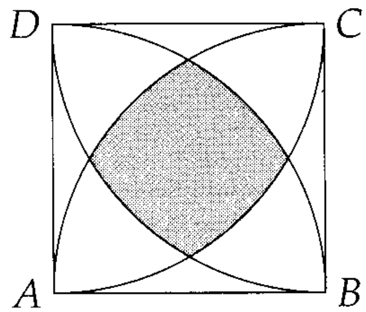
11. In the diagram below, XOY is a quarter-circle. Semicircles are drawn with diameters $\overline{OX}$ and $\overline{OY}$ as shown. Find the area of the shaded region given that $XO = 4$.



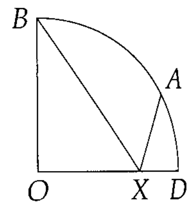
12. In the diagram below, $ABCD$ is a square of side length 4. Two quarter-circles and a diagonal are drawn as shown. Find the area of the shaded region.



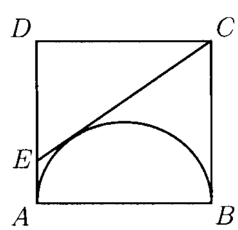
13. Quarter-circles are drawn centered at each vertex of square $ABCD$ as shown below. Given that $AB = 12$, find the area of the shaded region.



14. The figure below shows a quarter-circle of radius 1, with A on the arc BD such that $\angle AOD = 30°$. What must the distance OX be such that the region bounded by $\overline{AX}$, $\overline{BX}$, and the arc AB occupies half the area of the quarter-circle?

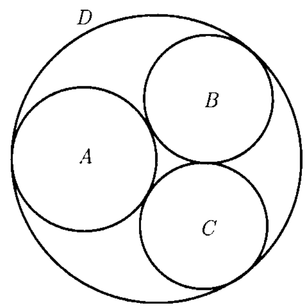


15. *(2004 AMC 10A, ♯22)* Square $ABCD$ has side length 2. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at E. What is the length of $\overline{CE}$?



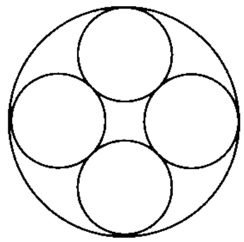
(A) $\dfrac{2+\sqrt{5}}{2}$ (B) $\sqrt{5}$ (C) $\sqrt{6}$ (D) $\dfrac{5}{2}$ (E) $5-\sqrt{5}$

16. *(2004 AMC 10A, ♯23)* Circles A, B, and C are externally tangent to each other and internally tangent to circle D. Circles B and C are congruent. Circle A has radius 1 and passes through the center of D. What is the radius of circle B?



(A) $\frac{2}{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{7}{8}$ (D) $\frac{8}{9}$ (E) $\frac{1+\sqrt{3}}{3}$

17. *(2009 AMC 10A, ♯21)* Many Gothic cathedrals have windows with portions containing a ring of congruent circles that are circumscribed by a larger circle. In the figure shown, the number of smaller circles is four. What is the ratio of the sum of the areas of the four smaller circles to the area of the larger circle?



(A) $3 - 2\sqrt{2}$ (B) $2 - \sqrt{2}$ (C) $4(3 - 2\sqrt{2})$ (D) $\frac{1}{2}(3 - \sqrt{2})$ (E) $2\sqrt{2} - 2$

18. *(2008-2009 USAMTS, Round 2, ♯2)* Let C_1, C_2, C_3, and C_4 be circles with radii 1, 1, 3, and 3, respectively, such that circles C_1 and C_2, C_2 and C_3, C_3 and C_4, and C_4 and C_1 are externally tangent. A fifth circle C is smaller than the four other circles and is externally tangent to each of them. Find the radius of C.

19. *(2005 AIME 1, $\sharp$1)* Six congruent circles form a ring with each circle externally tangent to the two circles adjacent to it. All six circles are internally tangent to a circle C with radius 30. Let K be the area of the region inside C and outside all of the six circles in the ring. Find $[K]$. (The notation $[K]$ denotes the greatest integer that is less than or equal to K).

20. *(2005 AIME 2, ♯8)* Circles C_1 and C_2 are externally tangent, and they are both internally tangent to circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively, and the centers of the three circles are collinear. A chord of C_3 is also a common external tangent of C_1 and C_2. Given that the length of the chord is $m\sqrt{n}/p$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

21. *(2002 AIME 1, ♯2)* The diagram shows twenty congruent circles arranged in three rows and enclosed in a rectangle. The circles are tangent to one another and to the sides of the rectangle as shown in the diagram. The ratio of the longer dimension of the rectangle to the shorter dimension can be written as $\frac{1}{2}(\sqrt{p} - q)$, where p and q are positive integers. Find $p + q$.