San Jose Math Circle February 7, 2009

CIRCLES AND FUNKY AREAS - PART I

DEFINITIONS:

- The set of all points that are the same distance from a given point is a *circle*. The given point is the *center* of the circle, and the fixed distance is the *radius*.
- We often refer to a circle by its center using the symbol $\odot$; $\odot O$ refers to a circle with center O.
- A line that touches a circle in a single point is *tangent* to the circle, while a line that hits a circle in two points is a *secant*. A segment connecting two points on a circle is a *chord*, and a chord that passes through the center of the circle is a *diameter*.
- The portion of a circle that connects two points on the circle is an *arc*, which we denote with the endpoints of the arc: $\overset{\frown}{MN}$ is the shorter arc that connects M and N.
- The perimeter of a circle is called the circle's *circumference*.
- A *central angle* of a circle is an angle whose vertex is at the center at the circle.
- A portion of a circle cut out by drawing two radii of the circle is called a *sector* of the circle.
- A portion of a circle between a chord and the arc of the circle connecting the endpoints of the chord is a *circular segment* of the circle.
- In every single circle, the ratio of circumference to diameter is the same. This ratio is called *pi*, and is given the symbol π. Its value is approximately 3.14. Pi is an irrational number, which means that is cannot be expressed as a ratio of integers. Because pi is irrational, its decimal expansion does not terminate and does not become periodic.

FUN FACTS:

- 22/7 is a common approximation of π. 355/113 is an even better approximation. In 1914, the great Indian mathematician Ramanujan provided the uncanny approximation:

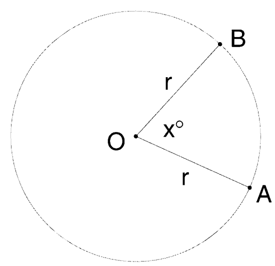
$$\pi \approx \sqrt[4]{9^2 + \frac{19^2}{22}}$$

- $\dfrac{\pi^2}{6} = \dfrac{1}{1^1} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2} + \ldots$

- In 1897, the Indiana state legislature almost passed a bill that set the value of π to exactly 3.2. The House voted unanimously for it and it passed a first reading in the Senate. Fortunately, a math professor at Purdue University happened to be visiting the legislature at the same time and advised that the bill be postponed indefinitely, effectively killing it.

FORMULAS:

Let $\odot O$ have radius r and let $\overline{AB}$ be a chord with $\angle AOB = x^\circ$.



- The degree measure of the arc $\overset{\frown}{AB}$ is:

$$\overset{\frown}{AB} = x^\circ$$

- The circumference C of the circle is:

$$C = 2\pi r$$

- The length of the arc $\overset{\frown}{AB}$ is:

$$\overset{\frown}{AB} = \frac{x}{360} \cdot 2\pi r$$

- The area A of the circle is:

$$A = \pi r^2$$

- The area of the sector AOB is:

$$\text{Area sector } AOB = \frac{x}{360} \cdot \pi r^2$$

REMARK. If the angle $\sphericalangle AOB$ is given in radians and $m(\sphericalangle AOB) = \theta$ radians, then the formulas for calculating the length of the arc $\overset{\frown}{AB}$ and the area of the sector AOB become:

- Length of the arc $\overset{\frown}{AB}$:

$$\overset{\frown}{AB} = r\theta$$

- Area of the sector AOB:

$$\text{Area sector } AOB = \frac{1}{2} r^2 \theta$$

1. *(Harvard-MIT Mathematics Tournament)* A man standing on a lawn is wearing a circular sombrero of radius 3 feet. Unfortunately, the hat blocks the sunlight so effectively that the grass directly under it dies instantly. If the man walks in a circle of radius 5 feet, what area of dead grass will result?

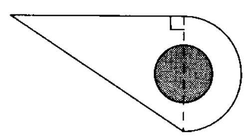
2. *(AOPS)* When my car has wheels with a diameter of 24 inches, its speedometer reports the correct speed of my car. I recently replaced my 24-inch wheels with 28-inch diameter wheels. I didn't change my speedometer, however. When the speedometer tells me that the car is going 40 miles per hour (and I'm driving with my 28-inch wheels), how fast is my car really going?

3. *(MATHCOUNTS)* Two congruent circular coins $\mathcal{A}$ and $\mathcal{Z}$ are touching at point P. $\mathcal{A}$ is held stationary while $\mathcal{Z}$ is rolled around it one time in such a way that the two coins remain tangent at all times. How many times will $\mathcal{Z}$ revolve around its center?

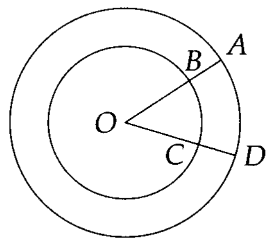
4. *(AOPS)* Regular hexagon $ABCDEF$ is inscribed in $\odot O$ with radius 6. What is the ratio of the circumference of the circle to the perimeter of the hexagon?

5. *(MATHCOUNTS)* On each side of a right triangle a semicircle is constructed using that side as a diameter. How many square centimeters are in the area of the semicircle on the hypotenuse of the right triangle if the areas of the semicircles on the legs of the triangle are 36 and 64 square centimeters?

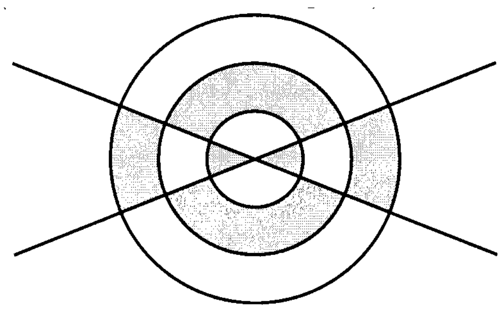
6. *(MATHCOUNTS)* What is the number of square centimeters in the area that is not shaded in the diagram below? The radius of the large semicircle is 1 centimeter, the radius of the small circle is 0.5 centimeters, and the length of the longer leg on the right triangle is 3 centimeters.



7. *(ARML)* The larger circle at right has radius 1.5 times the smaller circle. Compute the ratio of the partial ring $ABCD$ to the area of sector BOC.



8. *(2004 AMC 10A, ♯21)* Two distinct lines pass through the center of three concentric circles of radii 3, 2, and 1. The area of the shaded region in the diagram is 8/13 of the area of the unshaded region. What is the radian measure of the acute angle formed by the two lines? (Note: π radians is 180 degrees).



(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{7}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{5}$ (E) $\frac{\pi}{4}$

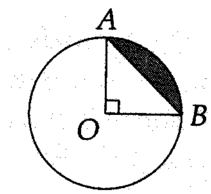
9. *(2000 AMC 10, ♯18)* Charlyn walks completely around the boundary of a square whose sides are exactly 5 km long. From any point on her path she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk, expressed in square kilometers and rounded to the nearest whole number?

(A) 24 (B) 27 (C) 39 (D) 40 (E) 42

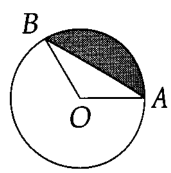
10. *(2003 AMC 10 A, ♯17)* The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of its circumscribed circle. What is the radius, in inches, of the circle?

(A) $\frac{3\sqrt{2}}{\pi}$ (B) $\frac{3\sqrt{3}}{\pi}$ (C) $\sqrt{3}$ (D) $\frac{6}{\pi}$ (E) $\sqrt{3}\pi$

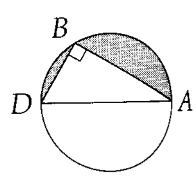
11. *(AOPS)* O is the center of the shown circle and $OA = 8$. The shaded region between chord $\overline{AB}$ and the circle is called a circular segment. Find the area of this circular segment.



12. *(AOPS)* Find the area of the shaded region given that O is the center of the circle, $\angle AOB = 120°$, and the radius of the circle is 6.

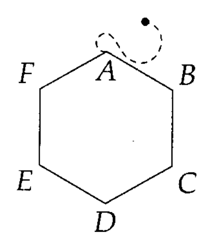


13. *(AOPS)* Given that $\triangle ABD$ in the diagram below is a right triangle with $BD = 8$ and $AB = 8\sqrt{3}$, find the total area of the shaded regions.



14. *(AOPS)* Each side of equilateral triangle XYZ has length 9. Find the area of the region inside the circumcircle of the triangle, but outside the triangle.

15. *(AOPS)* I have a barn that is a regular hexagon, as shown. Each side of the barn is 100 feet long. I tether my burro to point A with a 150 foot rope. Find the area of the region in which my burro can graze.



16. *(AOPS)* The shaded portion of the figure is called a *lune*. Given that $AB = 1$, $CD = \sqrt{2}$, and that $\overline{AB}$ and $\overline{CD}$ are diameters of the respective semicircles shown, find the area of the lune.