VIETE'S EQUATIONS - PART I

Viete's equations describe relationships between the zeros and the coefficients of a polynomial. They can be used for both polynomials with real coefficients, as well as for polynomials with complex coefficients.

Suppose that x_1 and x_2 are the zeros of the quadratic polynomial $p(x) = ax^2 + bx + c$. Then:

$$ax^2 + bx + c = a(x - x_1)(x - x_2) = ax^2 - a(x_1 + x_2)x + ax_1x_2$$

So $x_1 + x_2 = -\frac{b}{a}$ and $x_1x_2 = \frac{c}{a}$.

Similarly, if x_1, x_2, and x_3 are the zeros of the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$, we have:

$$\begin{cases} x_1 + x_2 + x_3 = -\frac{b}{a} \\ x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a} \\ x_1x_2x_3 = -\frac{d}{a} \end{cases}$$

In general, if $p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$ is a polynomial of degree n with zeros x_1, x_2, $\ldots$, x_n, then:

$$\begin{cases} x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n} \\ x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n = \frac{a_{n-2}}{a_n} \\ \vdots \\ x_1x_2 \ldots x_n = (-1)^n \frac{a_0}{a_n} \end{cases}$$

The equations above are the most frequently used; in general, the sum of the zeros taken i at a time, for $1 \le i \le n$ is given by:

$$x_1x_2 \ldots x_i + x_1x_2x_4 \ldots x_{i+1} + \cdots + x_{n-i+1}x_{n-i+2} \ldots x_n = (-1)^i \frac{a_{n-i}}{a_n}$$

1. *(1965 AMC 12, ♯7)* The sum of the reciprocals of the roots of the equation $ax^2+bx+c=0$ is:

(A) $\frac{1}{a}+\frac{1}{b}$ (B) $-\frac{c}{b}$ (C) $\frac{b}{c}$ (D) $-\frac{a}{b}$ (E) $-\frac{b}{c}$

2. Let x_1 and x_2 be the roots of the equation $x^2-\sqrt[3]{2}x-\sqrt[3]{4}=0$. Find $\frac{x_1}{x_2}+\frac{x_2}{x_1}$.

3. *(1988 AMC 12, ♯5)* Suppose that b and c are constants and

$$(x+2)(x+b) = x^2 + cx + 6$$

What is c?

(A) -5 (B) -3 (C) -1 (D) 3 (E) 5

4. *(1974 AMC 12, ♯2)* Let $x_1 \neq x_2$ be such that $3x_1^2 - hx_1 = b$ and $3x_2^2 - hx_2 = b$. What is $x_1 + x_2$?

(A) $-\frac{h}{3}$ (B) $\frac{h}{3}$ (C) $\frac{b}{3}$ (D) $2b$ (E) $-\frac{b}{3}$

5. *(2001 AMC 12, ♯19)* The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of the coefficients are all equal. The y-intercept of the graph of $y = P(x)$ is 2. What is b?

(A) -11 (B) -10 (C) -9 (D) 1 (E) 5

6. The solutions of the equation $x^2 + px + q = 0$ are the cubes of the solutions of the equation $x^2 + mx + n = 0$. Which of the following must be true?

(A) $p = m^3 + 3mn$ (B) $p = m^3 - 3mn$ (C) $p = 3mn - m^3$ (D) $p + q = m^3$

(E) $\left(\frac{m}{n}\right)^3 = \frac{p}{q}$

7. *(1961 AMC 12, ♯29)* Let the roots of $ax^2 + bx + c = 0$ be r and s. The equation with roots $ar + b$ and $as + b$ is:

(A) $x^2 - bx - ac = 0$ (B) $x^2 - bx + ac = 0$ (C) $x^2 + 3bx + ca + 2b^2 = 0$

(D) $x^2 + 3bx - ca + 2b^2 = 0$ (E) $x^2 + bx(2 - a) + a^2c + b^2(a + 1) = 0$

8. *(1962 AMC 12, ♯11)* The difference between the larger root and the smaller root of

$$x^2 - px + (p^2 - 1)/4 = 0$$

is:

(A) 0 (B) 1 (C) 2 (D) p (E) $p + 1$

9. *(1963 AMC 12, ♯14)* Consider the equations $x^2 + kx + 6 = 0$ and $x^2 - kx + 6 = 0$. If, when the roots of the equations are suitably listed, each root of the second equation is 5 more than the corresponding root of the first equation, then k equals:

   (A) 5 (B) -5 (C) 7 (D) -7 (E) none of these

10. *(2005 AMC 12, ♯12)* The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, and p is zero. What is the value of n/p?

    (A) 1 (B) 2 (C) 4 (C) 8 (E) 16

11. *(2005 AMC 12B, ♯17)* For some real numbers a and b, the equation:

$$8x^3 + 4ax^2 + 2bx + a = 0$$

has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of a?

(A) -256 (B) -64 (C) -8 (D) 64 (E) 256

12. *(2002 AMC 12A, ♯12)* Both roots of the equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is:

(A) 0 (B) 1 (C) 2 (D) 4 (E) more than four

13. *(2007 AMC 12A, ♯21)* The sum of the zeros, the product of the zeros, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Their common value must also be which of the following?

(A) the coefficients of x^2 (B) the coefficient of x (C) the y-intercept of the graph of $y = f(x)$ (D) one of the x-intercepts of the graph of $y = f(x)$

(E) the mean of the x-intercepts of the graph of $y = f(x)$

14. *(2002 AMC 12B, ♯6)* Suppose that a and b are nonzero real numbers and that the equation $x^2 + ax + b = 0$ has solutions a and b. Then the pair (a, b) is:

(A) $(-2, 1)$ (B) $(-1, 2)$ (C) $(1, -2)$ (D) $(2, -1)$ (E) $(4, 4)$