San Jose Math Circle January 17, 2009

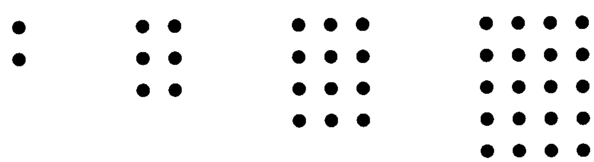
OBLONG, TRIANGULAR, AND SQUARE NUMBERS

One of the earliest subsets of natural numbers recognized by ancient mathematicians was the set of polygonal numbers. Such numbers represent an ancient link between geometry and number theory. Their origin can be traced back to the Greeks, where properties of oblong, triangular and square numbers were investigated and discussed by the sixth century BC pre-Socratic philosopher Pythagoras of Samos and his followers.

Pythagoras of Samos won a prize for wrestling at the Olympic Games at the age of 18. He studied with Thales, recognized as the father of Greek mathematics, and travelled extensively in Egypt and was well acquainted with Babylonian mathematics. At age 40, after teaching in Elis and Sparta, he migrated to Magna Graecia, where the Pythagorean school flourished at Croton in what is now southern Italy.

The most elementary class of polygonal numbers described by the early Pythagoreans was that of oblong numbers.

The nth oblong number: For $n \geqslant 1$, the nth oblong number, denoted by o_n, is given by $n(n+1)$ and represents the number of points in a rectangular array having n columns and $n+1$ rows.



The oblong numbers were generalized by Nicomachus of Gerasa (now Jerash in Jordan) in his number theoretic treatise "Introduction to Arithmetic". He defined the rectangular numbers $r_{n,k}$ as being numbers of the form $n(n+k)$, where $k \geqslant 1$ and $n > 1$.

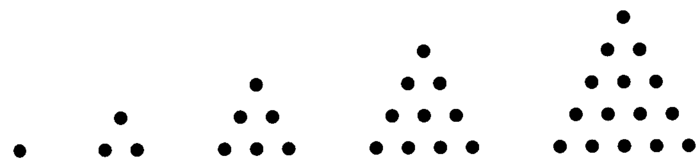
Note that $r_{n,1} = n(n+1)$ for all $n \geqslant 2$, so rectangular numbers do indeed generalize oblong numbers.

Nicomachus was one of the most important mathematicians of the ancient world; he was strongly influenced by Aristotle and is best known for his works Introduction to Arithmetic and The Manual of Harmonics in Greek. In Introduction to Arithmetic, Nicomachus writes extensively on numbers, especially on the significance of prime numbers and perfect numbers and argues that arithmetic is ontologically prior to the other mathematical sciences (geometry, music, and astronomy), and is their cause.

1. Show that if $p > 2$ is prime, then p^2 is not a rectangular number.

2. Write down the first ten rectangular numbers that are not oblong.

The triangular numbers 1, 3, 6, 10, 15, $\ldots$, t_n, $\ldots$, where t_n denotes the nth triangular number, represent the number of points used to portray an equilateral triangular pattern as shown:



The nth triangular number: For $n \geqslant 1$, the nth triangular number is defined to be the sum of the first n natural numbers. So $t_n = 1 + 2 + 3 + \cdots + n$.

Since the addition of natural numbers is both commutative and associative, we have:

$$t_n = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$t_n = n + (n-1) + (n-2) \cdots + 2 + 1$$

Adding columnwise we get $2t_n = \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ times}} =$

$= n(n+1)$. Hence

$$t_n = \frac{n(n+1)}{2}$$

Multiplying both sides of the latter equation by 2, we find that twice a triangular number is an oblong number:

$$2t_n = n(n+1) \Rightarrow 2t_n = o_n \text{ for all positive integers } n$$

3. Show that 40755 is a triangular number. [Ladies' Diary, 1828].

4. In 1991, S. P. Mohanty showed that there are exactly six triangular numbers that are the product of three consecutive integers. For example, $t_{20} = 210 = 5 \cdot 6 \cdot 7$. Show that t_{608} is the product of three consecutive positive integers.

5. Show that $9t_n+1$ [Fermat], $25t_n+3$ [Euler], and $49t_n+6$ [Euler] are triangular numbers.

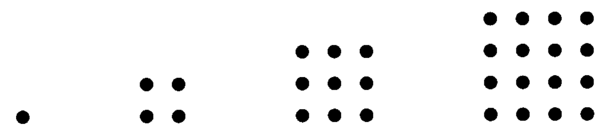
6. Generalize the previous exercise by showing that for all n, $m \geqslant 1$, the following equality holds:

$$(2m+1)^2 t_n + t_m = t_{(2m+1)n+m}$$

7. Let $\{t_n\}$ be a sequence defined by $t_1 = 1$ and $(2m+1)^2 t_n + t_m = t_{(2m+1)n+m}$ for all positive integers m and n. Show that t_n is the nth triangular number.

8. *(School Science and Mathematics, Problem 3831)* Prove that if t_n is the nth triangular number, then $(2k+1)^2 \cdot t_n + t_k$ is a triangular number.

The square numbers 1, 4, 9, 16, ... (sometimes also called "perfect squares") were represented geometrically by the Pythagoreans as square arrays of points:

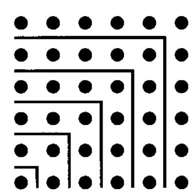


The nth square number: For $n \geqslant 1$, the nth square number, denoted by s_n, is defined by the formula $s_n = n^2$.

The Pythagoreans realized that the nth square number is the sum of the first n odd numbers, i.e.:

$$n^2 = 1 + 3 + 5 + \cdots + (2n - 1)$$

This property of the natural numbers first appears in Europe in Fibonacci's "Liber Quadratorum" (The Book of Squares). The $n = 6$ case is illustrated below:



Fibonacci's real name was Leonardo of Pisa. The nickname Fibonacci remains a mystery, but according to some sources Leonardo was "figlio de" Bonacci and thus known to us patronimically as Fibonacci. His father was a customs officer in Algeria, where Fibonacci learned to calculate with Hindu-Arabic numerals. He later helped popularize the Hindu-Arabic numeral system to Europe.

9. Prove that the nth square number exceeds its predecessor by the sum of the two roots. That is, prove that:

$$s_n - s_{n-1} = \sqrt{s_n} + \sqrt{s_{n-1}}$$

This appears in 1225 in Fibonacci's Liber Quadratorum.

10. Show that eight times a triangular number plus one is a square number. More precisely, show that:

$$8t_n + 1 = s_{2n+1}$$

This fact, known to the early Pythagoreans, appears in Plutarch's "Platonic Questions". Plutarch was a second century Greek biographer of noble Greeks and Romans. It is in his biography of Marcellus that we find one of the few accounts of the death of Archimedes during the siege of Syracuse, in 212 BC.

11. Prove algebraically that the sum of two consecutive triangular numbers is always a square number. Specifically, show that:

$$t_n + t_{n+1} = s_{n+1}$$

This property has been noted in the 2nd century by Theon of Smyrna in "On Mathematical Matters Useful for Reading Plato" and Nicomachus of Gerasa in "Introduction to Arithmetic". Theon demonstrates that every square $\geqslant 4$ is the sum of two consecutive triangular numbers geometrically, by drawing a line just above and parallel to the main diagonal of a square array.

12. Show that the difference between the squares of any two consecutive triangular numbers is always a cube. That is, show that:

$$t_{n+1}^2 - t_n^2 = (n+1)^3$$

13. Show that the product of any four consecutive natural numbers plus one is a square.

14. Find a positive integer $n > 1$ such that $1^2 + 2^2 + 3^2 + \cdots + n^2$ is a square number. [Ladies' Diary, 1792]

    This problem was posed by Edouard Lucas in 1875 in "Annales de Mathématique Nouvelle". In 1918, G. N. Watson proved that the problem has an unique solution.

15. Prove that $t_{2mn+m} = 4m^2 t_n + t_m + mn$, for every positive integers m and n.

16. Prove that $t_{m+n} = t_m + t_n + mn$ for all positive integers m and n.

17. Let $\{a_n\}$ be a sequence of such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$ for all positive integers m and n. Show that a_n is the nth triangular number.

18. Prove that the square of an odd multiple of 3 is the difference of two triangular numbers. More precisely, prove that:

$$[3(2n+1)]^2 = t_{9n+4} - t_{3n+1}$$

19. Show that there are infinitely many triangular numbers that are the sum of two triangular numbers.

Hint: $t_{[n(n+3)+1]/2} = t_{n+1} + t_{n(n+3)/2}$.

20. Calculate the sum of the first n oblong numbers. That is, find:

$$S = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1)$$

21. Calculate the sum of the reciprocals of the first n oblong numbers. That is, find:

$$S = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$$

22. Calculate the sum of the first n square numbers. That is, find:

$$S = 1^2 + 2^2 + 3^2 + \cdots + n^2$$