San Jose Math Circle December 20, 2008

INEQUALITIES - PART I

BASIC FACT. The square of any real number is a real number greater than or equal to 0.

The next few exercises make use of the above fact.

1. Show that for all real numbers x and y the following inequality holds:

$$x^2 + y^2 \geq 2xy$$

When does the equality occur?

2. Show that

$$x^3 + y^3 \geq x^2y + xy^2$$

for all positive real numbers x and y and deduce when the equality holds.

3. (i) Show that for all real numbers x the following inequalities hold:

$$x^2 + x + 1 > 0 \quad \text{and} \quad x^2 - x + 1 > 0$$

(ii) Show that for all real numbers x and y we have the inequalities:

$$x^2 + xy + y^2 \geq 0 \quad \text{and} \quad x^2 - xy + y^2 \geq 0$$

When do the equalities occur?

4. (i) Prove that

$$x^2 + y^2 + z^2 \geq xy + yz + zx$$

for all real numbers x, y, and z. When does the equality hold?

(ii) If x, y, and z are the lengths of the three sides of a triangle, then show that:

$$xy + yz + xz \leq x^2 + y^2 + z^2 < 2(xy + yz + xz)$$

(iii) Show that $\sqrt{6} + \sqrt{10} + \sqrt{15} < 10$, $\dfrac{1}{\sqrt{6}} + \dfrac{1}{\sqrt{10}} + \dfrac{1}{\sqrt{15}} < \dfrac{31}{30}$, and $3\sqrt{10} + 2\sqrt{15} + 5\sqrt{6} \leq 31$.

Let x and y be two positive numbers. We define:

- the arithmetic mean of x and y is the number $m_a = \dfrac{x+y}{2}$

- the geometric mean of x and y is the number $m_g = \sqrt{xy}$

- the harmonic mean of x and y is the number $m_h = \dfrac{2}{\dfrac{1}{x} + \dfrac{1}{y}}$

- the quadratic mean of x and y is the number $m_q = \sqrt{\dfrac{x^2 + y^2}{2}}$

In general, if x_1, x_2, $\ldots$, x_n are positive real numbers, then:

- the arithmetic mean of x_1, x_2, $\ldots$, x_n is the number $m_a = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$

- the geometric mean of x_1, x_2, $\ldots$, x_n is the number $m_g = \sqrt[n]{x_1 x_2 \ldots x_n}$

- the harmonic mean of x_1, x_2, $\ldots$, x_n is the number $m_h = \dfrac{n}{\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n}}$

- the quadratic mean of x_1, x_2, $\ldots$, x_n is the number $m_q = \sqrt{\dfrac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$

5. (*The Mean Inequalities*) Prove that for any positive numbers x and y:

$$\min\{x,y\} \le \frac{2}{\dfrac{1}{x}+\dfrac{1}{y}} \le \sqrt{xy} \le \frac{x+y}{2} \le \sqrt{\frac{x^2+y^2}{2}} \le \max\{x,y\}$$

and determine when equalities hold.

With the above notations,

$$\min\{x,y\} \le m_h \le m_g \le m_a \le m_q \le \max\{x,y\}$$

REMARK. We note that the Mean Inequalities generalize to n numbers:

$$\min\{x_1, x_2, \ldots, x_n\} \le m_h \le m_g \le m_a \le m_q \le \max\{x_1, x_2, \ldots, x_n\}$$

for all positive real numbers x_1, x_2, $\ldots$, x_n.

Of these inequalities, the arithmetic-geometric mean inequality is the most important and useful in applications:

$$\sqrt[n]{x_1 x_2 \ldots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all positive real numbers x_1, x_2, $\ldots$, x_n.

We will prove the cases $n = 3$ and $n = 4$ of the arithmetic-geometric mean inequality later in this handout.

6. (i) Prove that

$$x + \frac{1}{x} \geq 2$$

for all positive real numbers x. For what values of x does the equality hold?

(ii) Prove that

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

for all positive real numbers x and y. When does the equality hold?

7. Prove that

$$(x+y)(y+z)(z+x) \geq 8xyz$$

for all positive real numbers x, y, and z. When does the equality hold?

8. Prove that

$$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right) \geq 4$$

where x and y are positive real numbers. When does the equality hold?

9. Show that if x, y, and z are positive real numbers with $x+y+z=1$, then

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 9$$

10. Show that if a, b, and c are distinct positive real numbers, then the equation:

$$(a+b+c)x^2 - 6x + \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 0$$

has no real solutions.

11. (i) Show that for all real numbers x, y, and z the following holds:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

(ii) Deduce from (i) that if $x^3+y^3+z^3 = 3xyz$, then either $x+y+z = 0$ or $x = y = z$.

(iii) Prove that if x, y, and z are real numbers and $x + y + z \geq 0$, then

$$x^3 + y^3 + z^3 \geq 3xyz$$

(iv) Conclude using (iii) that if x, y, and z are positive real numbers we have:

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz}$$

This proves the arithmetic-geometric mean inequality in the case $n = 3$.

12. Show that

$$xy(x+y) + yz(y+z) + zx(z+x) \geq 6xyz$$

where x, y and z are positive real numbers and determine when equality holds.

13. Prove that if x, y, z are positive real numbers, then

$$\frac{x}{y} + \sqrt{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}} \geq \sqrt[3]{4} \cdot \sqrt{3}$$

14. Show that if x, y, z, and t are positive real numbers, then

$$\frac{x+y+z+t}{4} \geq \sqrt[4]{xyzt}$$

This proves the arithmetic-geometric mean inequality in the case $n = 4$.

15. Let x, y, z, and t be positive real numbers. Prove that:

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} \geq 4$$

16. If x_1, x_2, …, x_n are positive real numbers such that $x_1 x_2 \dots x_n = 1$, then

$$x_1^n + x_2^n + \cdots + x_n^n \geq n$$

17. If x_1, x_2, …, x_n are positive real numbers such that $x_1 x_2 \dots x_n = 1$, then

$$x_2 x_3 x_4 \dots x_n + x_1 x_3 x_4 \dots x_n + \cdots + x_1 x_2 \dots x_{n-1} \geq n$$

18. Let x_1, x_2, $\ldots$, x_n be positive real numbers. Prove the inequality:

$$\frac{(x_1^2 + x_1 + 1)(x_2^2 + x_2 + 1)\ldots(x_n^2 + x_n + 1)}{x_1 x_2 \ldots x_n} \geq 3^n$$

19. Let n be a positive integer. Show that: $1 \cdot 3 \cdot 5 \ldots (2n-1) < n^n$.

20. If a_1, a_2, ..., a_n are positive numbers with product equal to 1, then:

$$(1 + a_1)(1 + a_2) \dots (1 + a_n) \geq 2^n$$

21. Show that $\left(\frac{12}{5}\right)^5 + \left(\frac{12}{7}\right)^5 > 64$.

22. Find all real solutions of the following system of equations:

$$\begin{cases} 2x &= y + \dfrac{2}{y} \\ 2y &= z + \dfrac{2}{z} \\ 2z &= x + \dfrac{2}{x} \end{cases}$$