The Bouncing Ball Problem

And Other One-Player "Games"

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1. Imagine a pool table with pockets in the four corners. Represent it as a rectangle drawn on graph paper. Start with a ball at the bottom left corner, moving up at a 45° angle. The ball bounces off each side of the rectangle, always making a 45° angle with the sides, until finally it reaches one of the corners and falls in the pocket.

For example, draw a 5 by 10 pool table (width 5, height 10). The ball moves up and to the right, hits the middle of the right side of the table, moves up and to the left, and falls into the top left pocket.

a) How many times does the ball bounce on a 6 by 10 table? Which pocket does it land in?

b) Draw tables of various sizes and make a chart listing the dimensions of the table, how many times the ball bounces in total, how many of those bounces are on the left/right sides, how many are on the top/bottom sides, and which pocket the ball lands in. Make sure to include at least the 1 by 3, 1 by 4, 1 by 6, 3 by 5, 6 by 10, 9 by 15, 1 by 12, 2 by 12, 3 by 12, 4 by 12, 5 by 12, and 6 by 12 in your list, and as many others as you have time for.

c) Think about greatest common factor (and about writing and simplifying fractions). Your drawings should help you understand why some of the rows in the table give the same answers.

d) Using the tables you have drawn, can you predict which corner the ball will end in on a 79 by 103 table? 79 by 102? 78 by 102? Explain how you made your prediction. Can you prove your answer?

e) On each of those tables from part (d), predict how many top/bottom bounces and how many left/right bounces the ball will have. Again, explain how you made your prediction, then work on proving your answer.

f) To really understand this problem, symmetry is the key. It's an interesting kind of reflection symmetry, to be sure. But there's another symmetry: exchanging the ball and the table. What could I possibly mean by that?

2. [Thanks to A. Bogolmony of http://cut-the-knot.com for reminding me of this idea] In the game of squares and circles, begin with a collection of squares and circles. For example, you might start with three circles and a square. At each step, cross out any two shapes. If the shapes are the same, draw one square. If they are different, draw one circle. Eventually, there is only one shape left. You win if the final shape is a circle. What can you say about what happens in this game? What about with different starting situations?

3. In the previous problem, what if we have squares, circles, and triangles, and the rule is that you may only cross out two *different* shapes and then draw the third shape. Can you win (by ending with just one circle) if you start with three circles and a square? How about if you add the rule that you can cross out one shape and turn it into the other two (that is, the reverse of the usual rule)? What about with different starting situations?

4. Now, what if you cross out two different shapes and then draw *two* copies of the third shape? The goal is to get all the shapes the same. What starting situations enable you to eventually win?

5. The Mad Veterinarian [http://bumblebeagle.org/madvet/index.html, which incidentally also has some great solution discussions] has three machines. One converts a cat into two dogs and a mouse (or vice-versa): 1C <-> 2D 1M. A second machine does 1D <-> 1C 1M, and a third machine does 1M <-> 1C 3D. The general puzzle is to start with just one animal and replicate it: what's the fewest cats (more than one) that you can turn one cat into (with no mice or dogs left around)?

6. Here's a two-player game for a change: Player 1 writes a sequence of ten positive integers. Then player 2 decides who goes first. Then the players take turns writing either a + or – sign in the spaces between the integers. In the end, if the player who moved last makes the result even, they win, otherwise the other player wins.

7. Another one-player game: Start with a stack of n boxes. At each move, as long as any stacks have more than one box, split one stack into two parts, say x boxes into y and z, and score yz points. How should you split them in order to maximize your score? What is the maximum score for each n?

8. Coin-flipping: Begin with some number of coins, say four for example, and set them on the table in a line, with a given starting sequence like HHTH for example. At each move, you may flip any two adjacent coins. You win if the final arrangement of the coins is all heads.

9. Coin-splitting: Begin with an infinite strip of squares, and a penny on one spot. At each move, you may either split the penny (remove it and put a penny on each adjacent spot) or merge two pennies (remove two pennies with a space between them and put one on the space between). You may have any number of pennies on a given spot. Starting with one penny, can you split and merge to end up with just one penny on the board in a different spot? What different spots are possible?