Wrangle 2012 – Practice 3

1. By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself. A natural number greater than 1 will be called "nice" if it is equal to the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

2. Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

3. Prove that every scalene triangle contains two sides such that the quotient of their lengths is less than or equal to 1 but greater than 3/5.

4. Suppose we start with an $n \times n$ square consisting of n^2 white cells. We can pick any cell and then paint black this cell, all cells that share an edge with this cell, and all cells in the same column with the chosen cell. We can repeat this process any number of times; it's allowed to paint a cell black more than once. What is the least number of times this process should be repeated so that all n^2 cells become black?

5. Suppose that points A, B, C are vertices of a scalene triangle. How many points D in the plane of ΔABC have the property that quadrilateral $ABCD$ has at least one axis of symmetry?

6. Find the number of solutions of the given equation for various values of the parameter b:
$$\sqrt{3x-5} = b - \sqrt{3x+11}.$$

7. Let S be the sum of the base 10 logarithms of all the proper divisors (all divisors of a number excluding itself) of 1,000,000. What is the integer nearest to S?

8. Consider the region A in the complex plane that consists of all points z such that both $\frac{z}{40}$ and $\frac{40}{\bar{z}}$ have real and imaginary parts between 0 and 1, inclusive. What is the area of A?